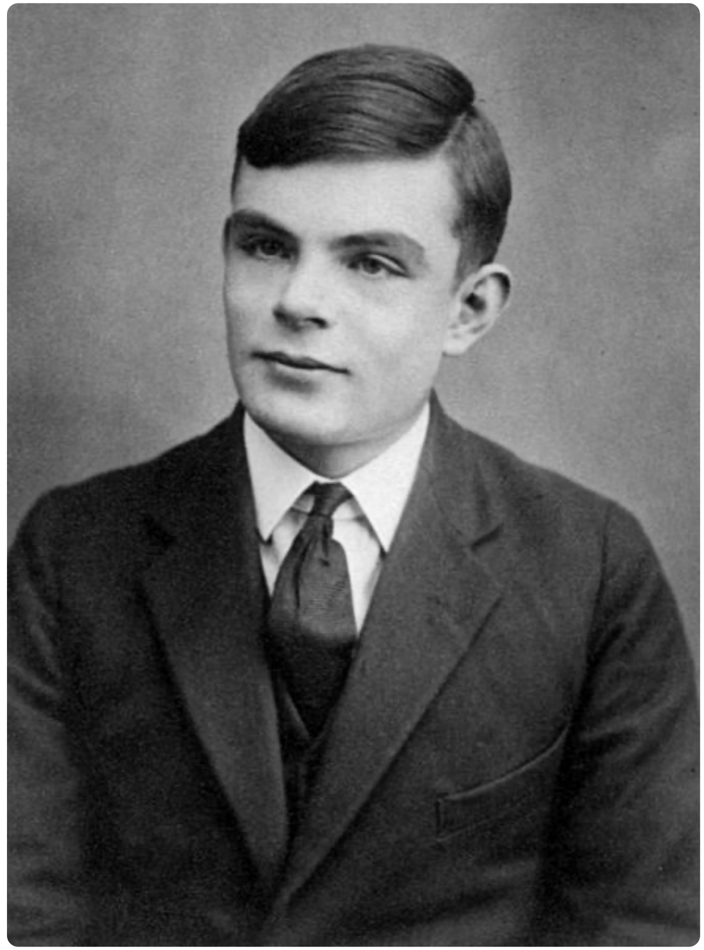
Ada Lovelace



Alan Turing

welcome to Graduate Algorithms

fundamentalalgorithms.com

Graduate Algorithms, CS580, Spring 2024

Lectures: 11:30AM to 12:20 PM; on Monday, Wednesday and Friday; in WALC 3087. *First two lectures are over Zoom.*

one big .pdf w/ everything

Staff:	Email (@purdue.edu)	Office hours
Kent Quanrud	`krq`	TBD at LWSN 1211[1]
Tiantian Gong (TA)	`gong146`	Tuesday, 11AM–12PM, at HAAS 143.
Haoyu Song (TA)	`song522`	Monday, 3–4PM, at HAAS 143.

Please see the syllabus (page 549) for more information.

Homework

- Homework 0 due January 17.
- Homework 1 due January 24.
- Homework 2 due January 31.
- Homework 3 due February 7.
- Homework 4 due February 21.
- Homework 5 due February 28.
- Homework 6 due March 6.
- Homework 7 due March 20.
- Homework 8 due April 3.
- Homework 9 due April 10.
- Homework 10 due April 17.

Class schedule[2]

1. *January 8.* Searching and sorting, including binary search, merge sort, and lower bounds for sorting (chapter 1). *On zoom:* `https://purdue-edu.zoom.us/j/98283084796?pwd=MEE1RlhSdi9BUksyY3lCZWhzcXpNZz09`

welcome to Graduate Algorithms

fundamentalalgorithms.com

Graduate Algorithms, CS580, Spring 2024

Lectures: 11:30AM to 12:20 PM; on Monday, Wednesday and Friday; in WALC 3087. *First two lectures are over Zoom.*

Staff:	Email (`@purdue.edu`)	Office hours
Kent Quanrud	`krq`	TBD at LWSN 1211[1]
Tiantian Gong (TA)	`gong146`	Tuesday, 11AM–12PM, at HAAS 143.
Haoyu Song (TA)	`song522`	Monday, 3–4PM, at HAAS 143.

Please see the syllabus (page 549) for more information.

Homework

- Homework 0 due January 17.
- Homework 1 due January 24.
- Homework 2 due January 31.
- Homework 3 due February 7.
- Homework 4 due February 21.
- Homework 5 due February 28.
- Homework 6 due March 6.
- Homework 7 due March 20.
- Homework 8 due April 3.
- Homework 9 due April 10.
- Homework 10 due April 17.

Class schedule[2]

1. *January 8.* Searching and sorting, including binary search, merge sort, and lower bounds for sorting (chapter 1). *On zoom:* `https://purdue-edu.zoom.us/j/98283084796?pwd=MEE1RlhSdi9BUksyY3lCZWhzcXpNZz09`

Please read the syllabus

- will ask for questions next class

Highlights

Awesome TA's

2 midterms, final

weekly homework

drop lowest 20% of HW scores

late/resubmit HW policy

Part I.

You already know TCS

- counting
- over/under

Counting

Counting

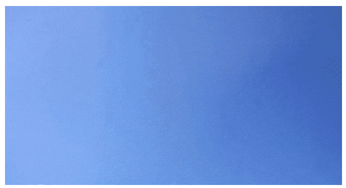
||||||||||||||||||||||| = 23

RHS much more efficient than LHS

"unary"

Decimal notation: 10^k #'s w/ k digits

Combinatorial explosion!



Over/under

guess # between 1 and 100

Slow algo:

for $i = 1, 2, \cdots, 100$, guess i

FAST

50

under over

25 75

under over under over

12 38 63 88

rounds?

$\log_2 100 \approx 6.67$

$\Rightarrow$ 6 rounds

exponentials	logarithms
$2^n = \underbrace{2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2}_{n \text{ times}}$	"logarithm (base 2) of n" $\log_2 n$ s.t. $2^{\log_2 n} = n$ also: $\log_e n$ (aka $\ln$) $\log_{10} n$ $\log_b n$ for some $b>1$

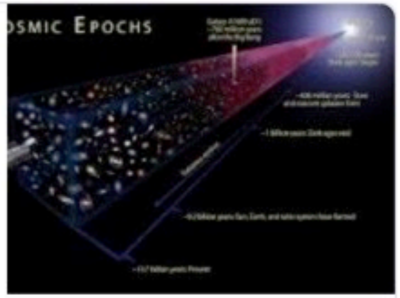
Google

how many atoms in the universe

All | Images | News | Shopping | Videos | More | Settings | Tools

About 26,300,000 results (0.49 seconds)

10^{82} atoms



At this level, it is estimated that the there are between 10^{78} to 10^{82} **atoms** in the known, observable **universe**. In layman's terms, that works out to between ten quadrillion vigintillion and one-hundred thousand quadrillion vigintillion **atoms**. Jul 30, 2009

How Many Atoms Are There in the Universe? - Universe Today

https://www.universetoday.com › atoms-in-the-universe

$$\log_2(\#\,\text{atoms}) = \log_2(10^{82}) = 82\underbrace{\log_2(10)}_{\approx 3.32}$$

$$\approx 272.398$$

$$\log_2(\log_2(\#\,\text{atoms})))$$

$$\approx \log_2(272.4)$$

$$\approx 8.\sim$$

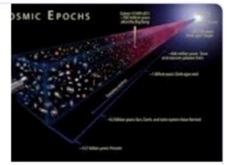
$$(2^8 = 256)$$

how many atoms in the universe

All Images News Shopping Videos More Settings Tools

About 26,300,000 results (0.49 seconds)

10^{82} atoms



At this level, it is estimated that the there are between 10^{78} to 10^{82} **atoms** in the known, observable **universe**. In layman's terms, that works out to between ten quadrillion vigintillion and one-hundred thousand quadrillion vigintillion **atoms**. Jul 30, 2009

How Many Atoms Are There in the Universe? - Universe Today
https://www.universetoday.com › atoms-in-the-universe

$$\log_2(\#\,\text{atoms})$$

$$= \log_2(10^{82})$$

$$= 82\log_2(10)$$

$$\approx 272.398$$

TCS is about distinguishing

2^n, combinatorial explosion vs $\log_2 n$, binary search

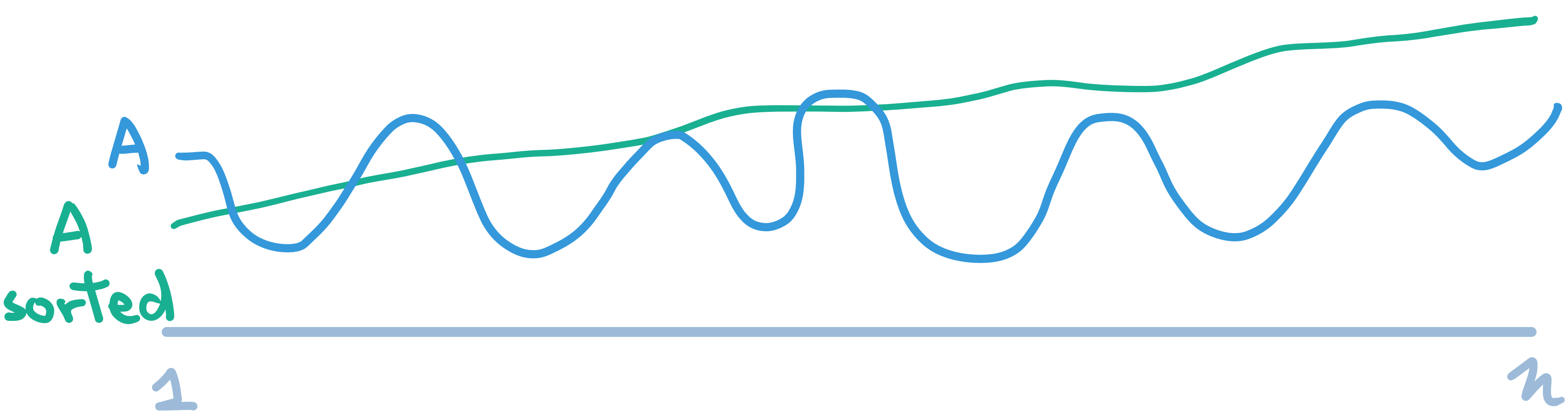
Part II: Sorting

Input: n numbers

5, 1, 8, 9, 2, 6, 7, 4, 11, 7, 9

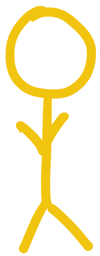
Goal: sorted list

1, 2, 4, 5, 7, 7, 8, 9, 9, 11



Human sort

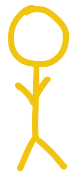
5, $\boxed{1,}$ 8, 9, $\boxed{2,}$ 6, 7, 4, 11, 7, 9



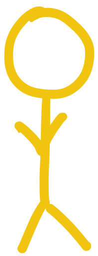
- what comes first?
- what comes second?

1 2

5, $\boxed{1}$, 8, 9, $\boxed{2}$, 6, 7, 4, 11, 7, 9



- what comes first?
- what comes second?



builds sorted list from beginning to end.

each time, we scan the list of n inputs

$$
\begin{array}{ll}
 & n \text{ iterations} \\
\times & n \text{ items scanned per iteration} \\
\hline
\approx & n^2 \text{ time}
\end{array}
$$

Big-O

$$\begin{array}{ll} & n \text{ iterations} \\ \times & n \text{ items scanned per iteration} \\ \hline & n^2 \text{ time} \end{array}$$

"$O(n^2)$": there exists constants $N, c > 0$ s.t. for all $n > N$, the algorithm terminates in at most cn^2 steps

WTF "step"?! varies by computer, language, ~

low-level details only effect the constant

$O(\sim)$ hides the constant so we can develop a general THEORY OF ALGORITHMS

Human sort: $O(n^2)$ time

(aka selection sort)

Better?

n^2 = each # is compared to every other #

can we sort all #'s without comparing every pair?

do we really need all comparisons?

Slightly different perspective

given array $A[1 \sim n]$ that is supposedly sorted

how long does it take to verify A is sorted?

$$a_1 \overset{\leq ?}{\wedge} a_2 \overset{\leq ?}{\wedge} a_3 \quad a_4 \quad a_5 \quad a_6 \quad \sim$$

$O(n)$

sorted?($A[1 \sim n]$)

1. for $i = 1, \ldots, n-1$
   a. if $A[i] > A[i+1]$
      1. return False
2. return True

Human sort: $O(n^2)$ time

(aka selection sort)

Verifying a sort: $O(n)$

Better?

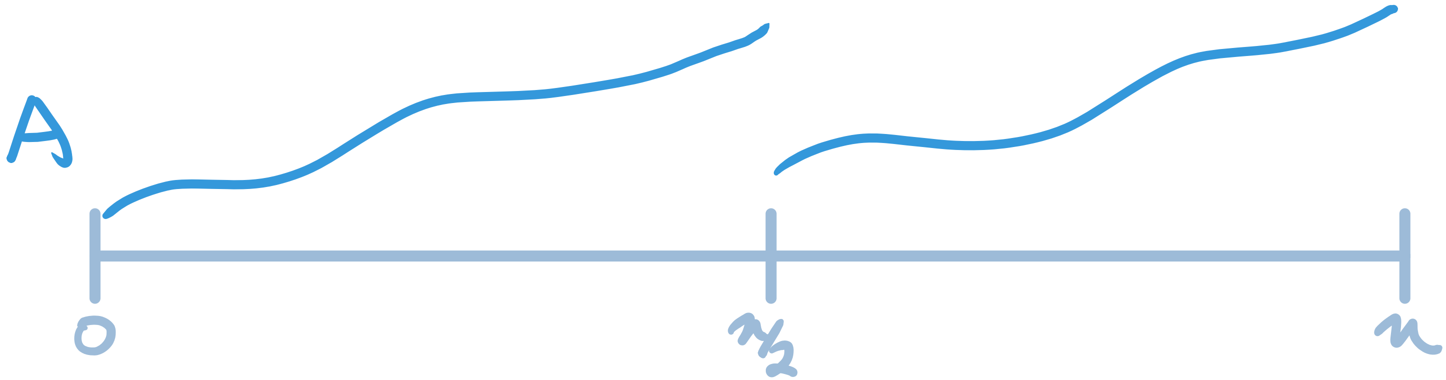
n^2 = each # is compared to every other #

can we sort all #'s without comparing every pair?

do we really need all comparisons?

Simple case

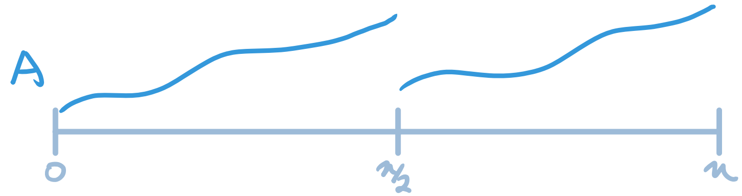
suppose first & second halves are already sorted



1, 2, 5, 6, 8, 9,   4, 7, 7, 9, 11

Simple case

suppose first & second halves are already sorted



1, 2, 5, 6, 8, 9, 4, 7, 7, 9, 11

merge($A[1..m]$, $B[1..n]$)

/ Given two sorted lists A and B, produces the combined lists in sorted order. */*

1. Allocate a new array $C[1..m+n]$, let $i = 1$, and let $j = 1$.
2. While $i \leq m$ and $j \leq n$:
   A. If $A[i] \leq B[j]$:
      1. $C[i+j-1] \leftarrow A[i]$ and $i \leftarrow i+1$.
   B. Else $(B[j] < A[i])$:
      1. $C[i+j-1] \leftarrow B[j]$ and $j \leftarrow j+1$.
3. While $i \leq m$:
   A. $C[i+j-1] \leftarrow A[i]$ and $i \leftarrow i+1$.
4. While $j \leq n$:
   A. $C[i+j-1] \leftarrow B_2[j]$ and $j \leftarrow j+1$.
5. Return C.

merge($A[1..m]$, $B[1..n]$)

/ Given two sorted lists A and B, produces the combined lists in sorted order.*

1. Allocate a new array $C[1..m+n]$, let $i = 1$, and let $j = 1$.
2. While $i \leq m$ and $j \leq n$:
   - A. If $A[i] \leq B[j]$:
     - 1. $C[i+j-1] \leftarrow A[i]$ and $i \leftarrow i+1$.
   - B. Else ($B[j] < A[i]$):
     - 1. $C[i+j-1] \leftarrow B[j]$ and $j \leftarrow j+1$.
3. While $i \leq m$:
   - A. $C[i+j-1] \leftarrow A[i]$ and $i \leftarrow i+1$.
4. While $j \leq n$:
   - A. $C[i+j-1] \leftarrow B_2[j]$ and $j \leftarrow j+1$.
5. Return C.

Divide-and-conquer

1. Sort first half recursively
2. Sort second half recursively
3. merge sorted halves together

Designing a recursive algorithm

Divide-and-conquer
1. Sort first half recursively
2. Sort second half recursively
3. merge sorted halves together

① Recursive specification

`merge-sort`($A[1..n]$): Given an array $A[1..n]$ of comparable elements, returns an array consisting of the elements of A sorted in increasing order.

- declares input/output of algo
- "contract" algo must fulfill
- induction hypothesis for proof of correctness

Designing a recursive algorithm

Divide-and-conquer
1. Sort first half recursively
2. Sort second half recursively
3. merge sorted halves together

① Recursive specification

`merge-sort`$(A[1..n])$: Given an array $A[1..n]$ of comparable elements, returns an array consisting of the elements of A sorted in increasing order.

- declares input/output of algo
- "contract" algo must fulfill
- induction hypothesis for correctness

② Implement spec

`merge-sort`$(A[1..n])$

/ In the base case, the list is so short there is nothing to do. */*

1. If $n \leq 1$ then return A.

Base case

/ In the general case, we divide the input in half and recursively sort each half. */*

2. Let $m = \left\lfloor \frac{n}{2} \right\rfloor$.
3. $B_1[1..m] \leftarrow \text{merge-sort}(A[1..m])$.
4. $B_2[1..n-m] \leftarrow \text{merge-sort}(A[m+1..n])$.

satisfies recursive spec by induction on n

/ Now we merge the two sorted halves in linear time. */*

5. return `merge`(B_1, B_2).

Designing a recursive algorithm

Divide-and-conquer

1. Sort first half recursively
2. Sort second half recursively
3. merge sorted halves together

① Recursive specification

`merge-sort`($A[1..n]$): Given an array $A[1..n]$ of comparable elements, returns an array consisting of the elements of A sorted in increasing order.

- declares input/output of algo
- "contract" algo must fulfill
- induction hypothesis for correctness

② Implement spec

<u>`merge-sort`($A[1..n]$)</u>

/ In the base case, the list is so short there is nothing to do. */*

1. If $n \leq 1$ then return A. ← Base case

/ In the general case, we divide the input in half and recursively sort each half. */*

2. Let $m = \lfloor \frac{n}{2} \rfloor$.
3. $B_1[1..m] \leftarrow$ merge-sort($A[1..m]$).
4. $B_2[1..n-m] \leftarrow$ merge-sort($A[m+1..n]$).

satisfies recursive spec by induction on n

/ Now we merge the two sorted halves in linear time. */*

5. return `merge`(B_1, B_2).

③ Prove correctness

argue algo satisfies recursive spec for all n by induction on n

Divide-and-conquer

1. Sort first half recursively
2. Sort second half recursively
3. merge sorted halves together

`merge-sort`$(A[1..n])$: Given an array $A[1..n]$ of comparable elements, returns an array consisting of the elements of A sorted in increasing order.

`merge-sort`$(A[1..n])$

/ In the base case, the list is so short there is nothing to do. */*

1. If $n \leq 1$ then return A.

Base case

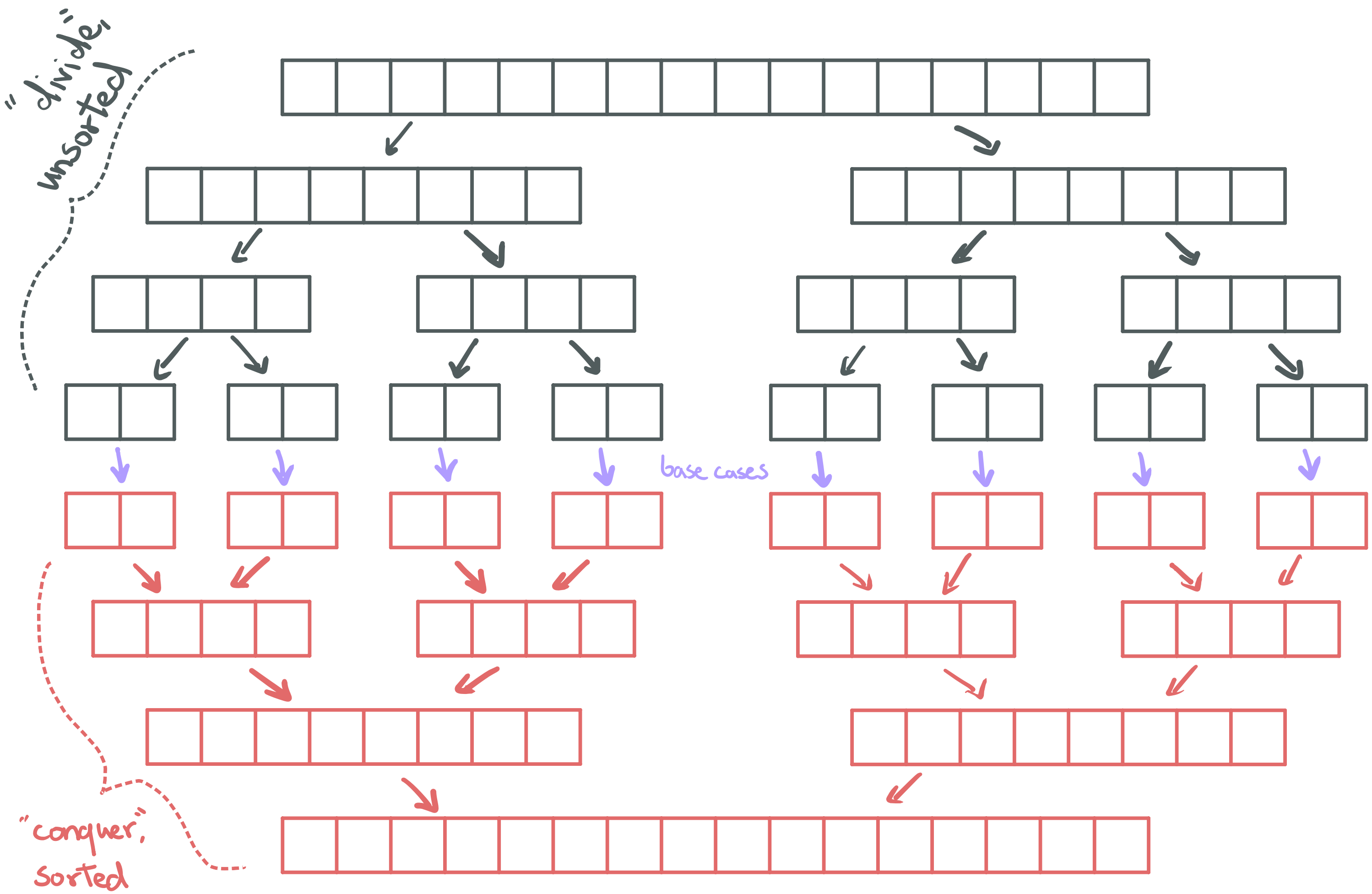
/ In the general case, we divide the input in half and recursively sort each half. */*

2. Let $m = \lfloor \frac{n}{2} \rfloor$.
3. $B_1[1..m] \leftarrow$ merge-sort$(A[1..m])$.
4. $B_2[1..n-m] \leftarrow$ merge-sort$(A[m+1..n])$.

satisfies recursive spec by induction on n

/ Now we merge the two sorted halves in linear time. */*

5. return `merge`(B_1, B_2).



Running time analysis

$T(n)$ = time for input of size n

$T(1) = T(0) = 1$

$n > 1$:

$$T(n) = T\left(\left\lceil \tfrac{n}{2} \right\rceil\right) + T\left(n - \left\lceil \tfrac{n}{2} \right\rceil\right) + O(n)$$

$$\overset{*}{\approx} 2T\left(\tfrac{n}{2}\right) + O(n)$$

Divide-and-conquer

1. Sort first half recursively
2. Sort second half recursively
3. merge sorted halves together

`merge-sort`$(A[1..n])$: Given an array $A[1..n]$ of comparable elements, returns an array consisting of the elements of A sorted in increasing order.

`merge-sort`$(A[1..n])$

/* In the base case, the list is so short there is nothing to do. */

1. If $n \leq 1$ then return A. ← Base case

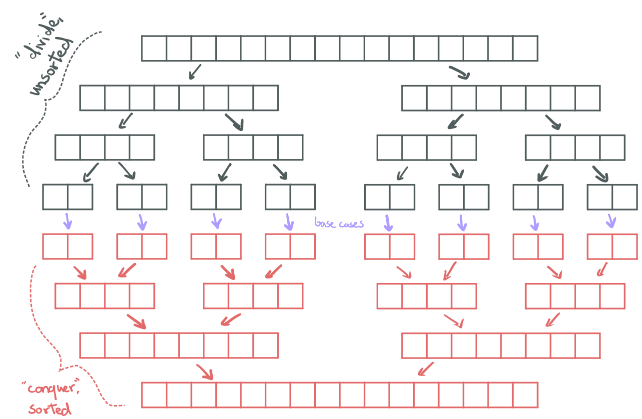
/* In the general case, we divide the input in half and recursively sort each half. */

2. Let $m = \left\lfloor \frac{n}{2} \right\rfloor$.
3. $B_1[1..m] \leftarrow$ merge-sort$(A[1..m])$.
4. $B_2[1..n-m] \leftarrow$ merge-sort$(A[m+1..n])$.

satisfies recursive spec by induction on n

/* Now we merge the two sorted halves in linear time. */

5. return `merge`(B_1, B_2).

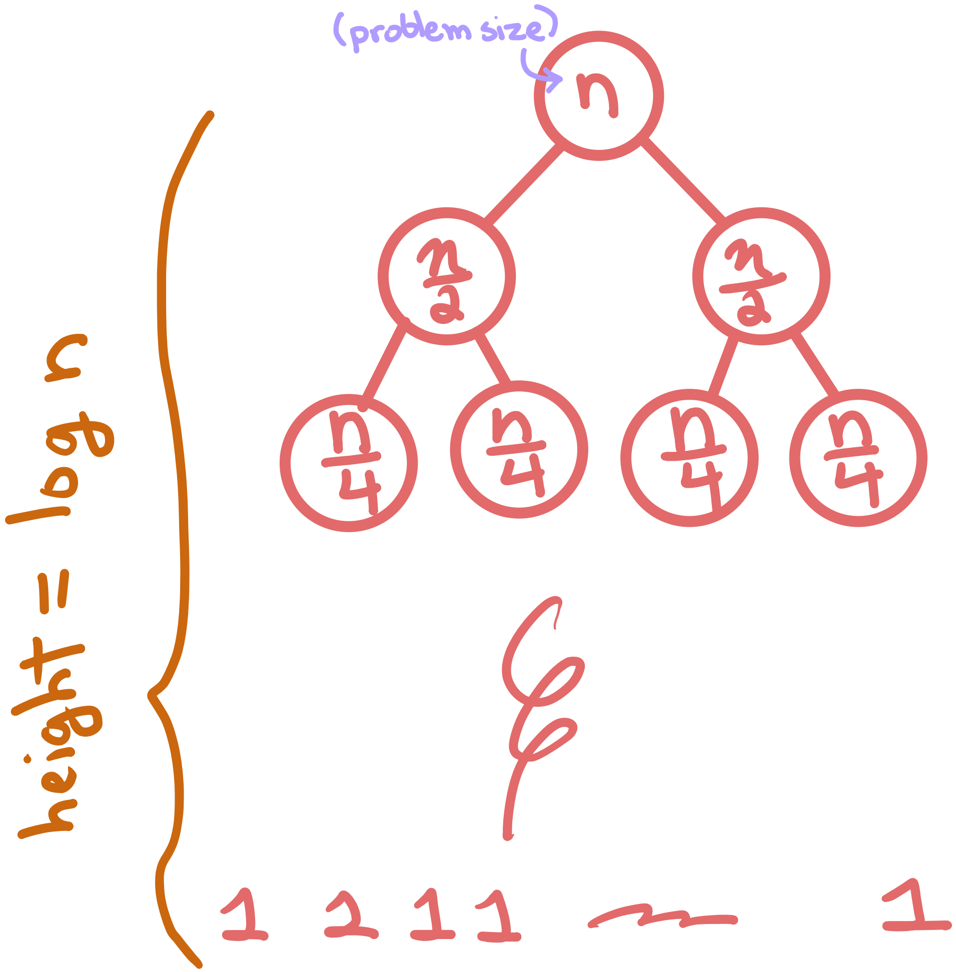


* see notes for justification

Recursion tree

$T(1)=T(0)=1$

$T(n)=2T(\frac{n}{2})+O(n)$



Work

n

$2\cdot(\frac{n}{2})=n$

$4\cdot(\frac{n}{4})=n$

$\vdots$

$2^k\cdot\frac{n}{2^k}=n$

$+\ n\cdot 1=n$

$n\times\log(n)$

$\Rightarrow T(n)=O(n\log n)$

III

Lower Bounds for Sorting

Lower Bounds for Sorting

merge-sort: $O(n \log n)$. better? optimal?

Goal: prove lower bound for all sorting algorithms

e.g. prove any correct algorithm has to take at least

$\underbrace{\Omega(f(n))}_{\text{"big-Omega"}}$ time

(like $O(\sim)$ but for lower bounds)

Lower Bounds for Sorting
merge-sort: $O(n \log n)$. better? optimal?
Goal: prove lower bound for all sorting algorithms
e.g. prove any correct algorithm has to take at least
$\underbrace{\Omega(f(n))}_{\text{"big-Omega"}}$ time
(like $O(\cdot)$ but for lower bounds)

Comparison model

- Input: $x_1, \ldots, x_n$ comparable
- only info is via comparison queries
  "is $x_i < x_j$?"

Goal lower bound # comparison Q's

Suppose algo makes k queries and outputs list $(x_{i_1}, x_{i_2}, \ldots, x_{i_n})$

Comparison model

- Input: $x_1, \ldots, x_n$ comparable
- only info via comparison queries "is $x_i < x_j$?"

<u>Goal</u> lower bound # comparisons

1. $x_{i_1} < x_{j_1}$?
2. $x_{i_2} < x_{j_2}$?
3. $x_{i_3} < x_{j_3}$?
4. $x_{i_4} < x_{j_4}$?

$\vdots$

k. $x_{i_k} < x_{j_k}$?

$\Rightarrow (x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_n})$

- output is a function of k queries
- each query has 2 outcomes (YES/NO)

$\Rightarrow$ at most 2^k possible outputs meanwhile...

$n!$ possible lists

$\Rightarrow 2^k \geq n!$ just to be able to output all lists

$$k \geq \underbrace{\log(n!)}$$

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$$
$$\geq \left(\frac{n}{2}\right)^{n/2}$$

Comparison model

- Input: $x_1, \ldots, x_n$ comparable
- only info via comparison queries
  "is $x_i < x_j$?"

Goal lower bound # comparisons

Suppose algo makes k queries and outputs list $(x_{i_1}, x_{i_2}, \ldots, x_{i_n})$

1 $x_{i_1} < x_{j_1}$?
2 $x_{i_2} < x_{j_2}$?
3 $x_{i_3} < x_{j_3}$?
4 $x_{i_4} < x_{j_4}$?
⋮
k $x_{i_k} < x_{j_k}$?

$\Rightarrow (x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_n})$

- output is a function of k queries
- each query has 2 outcomes (YES/NO)

$\Rightarrow$ at most 2^k possible outputs

$$k \geq \log\left(\left(\frac{n}{2}\right)^{n/2}\right) = \frac{n}{2}\log\left(\frac{n}{2}\right) = \Omega(n \log n)$$

Theorem In the comparison (only) model, any (correct, determistic) sorting algorithm makes

$$\Omega(n \log n) \text{ comparisons}$$

in the worst case.

Upper bound	Lower bound
$O(n \log n)$ (mergesort)	$\Omega(n \log n)$

in general (any lower bound) $\le$ (any upper bound)

here we also have

(an upper bound) $\le$ (some constant) (a lower bound)

[$O(n\log n)$] [$\Omega(n\log n)$]

bounds mutually certify each other to be optimal

"bounds are tight" (up to constants)

Conclusion: $n\log n$ is "right answer" for sorting in comparison model

Graduate Algorithms, CS580, Spring 2024

Lectures: 11:30AM to 12:20 PM; on Monday, Wednesday and Friday; in WALC 3087. *First two lectures are over Zoom.*

Staff:	Email (`@purdue.edu`)	Office hours
Kent Quanrud	`krq`	TBD at LWSN 1211[1]
Tiantian Gong (TA)	`gong146`	Tuesday, 11AM–12PM, at HAAS 143.
Haoyu Song (TA)	`song522`	Monday, 3–4PM, at HAAS 143.

Please see the syllabus (page 549) for more information.

Homework

- Homework 0 due January 17.
- Homework 1 due January 24.
- Homework 2 due January 31.
- Homework 3 due February 7.
- Homework 4 due February 21.
- Homework 5 due February 28.
- Homework 6 due March 6.
- Homework 7 due March 20.
- Homework 8 due April 3.
- Homework 9 due April 10.
- Homework 10 due April 17.

Class schedule[2]

1. *January 8.* Searching and sorting, including binary search, merge sort, and lower bounds for sorting (chapter 1). *On zoom:* `https://purdue-edu.zoom.us/j/98283084796?pwd=MEE1RlhSdi9BUksyY3lCZWhzcXpNZz09`
2. *January 10.* Induction and recursion (chapter 2). *On zoom:* `https://purdue-edu.zoom.us/j/93963400762?pwd=ZVY1bXd4MGVKVDdjZnlESkh2N2Izdz09`
3. *January 12.* Catch-up day.

[1] Office hours canceled on the week of January 8 due to travel.
[2] All future dates are tentative. Lecture materials and video recordings can be found at the end of each chapter.