Selection & Closest Pair

Selection

Input: n comparable elem. $A[1 \to n]$, $k \in [n]$

Goal: kth smallest element of $A[1..n]$

e.g. median ($k = \lceil n/2 \rceil$) of

10, 56, 77, 91, 70, 24, 36, 27, 64, 50, 65

Obv. approach: sort A, look up kth smallest

$O(n \log n)$

Better?

① Recursive specification

Designing a recursive algorithm
① Recursive specification
- declares input/output of algo
- "contract" algo must fulfill
- induction hypothesis for correctness

② Implement spec

③ Prove correctness
argue algo satisfies recursive spec for all n by induction on n

`select`$(k, A[1..n])$: Given an array $A[1..n]$ of comparable elements and an integer $k \in \{1, \ldots, n\}$, returns the kth smallest element out of $A[1..n]$.

① Recursive specification

`select`$(k, A[1..n])$: Given an array $A[1..n]$ of comparable elements and an integer $k \in \{1, \ldots, n\}$, returns the kth smallest element out of $A[1..n]$.

② Implement spec

thought experiment: suppose O(n)-time subroutine

$$\texttt{median}(A[1..n]) = \texttt{select}(\lceil n/2 \rceil, A[1..n])$$
$$= \text{the } \lceil n/2 \rceil \text{th smallest element out of } A[1..n].$$

how to use median to find kth element?

"pivot"

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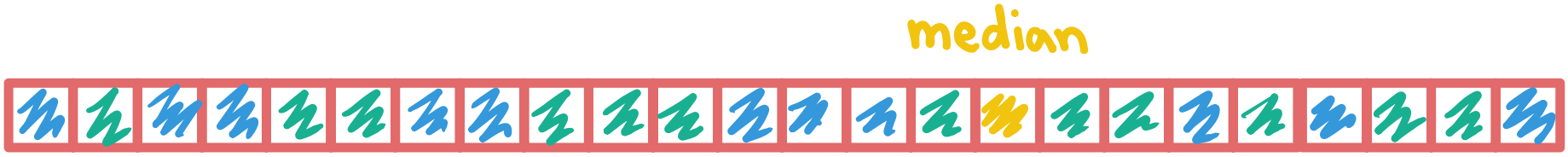
① find median



② divide input

③ recurse on appropriate half

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`select`$(k, A[1..n])$

/ `select`$(k, A[1..n])$ = the kth smallest element in $A[1..n]$. We assume the elements in A are distinct for simplicity. (Otherwise, break ties arbitrarily.) */*

1. If $n \leq 10$ then find the kth element by brute force.
2. Let $p =$ `median`(A). *// We assume a linear time subroutine for* `median`$(A[1..n])$.
3. If $k = \lceil n/2 \rceil$ then return p.
4. If $k < \lceil n/2 \rceil$ then return `select`(k, B), where $B[1..\lceil n/2 \rceil - 1]$ is an array of the $\lceil n/2 \rceil - 1$ elements smaller than p.
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Running time

$$T(n) = T(n/2) + O(n)$$

	work
n	n
$\vdots$	
$n/2$	$n/2$
$\vdots$	
$n/4$	$n/4$
$\vdots$	
1	1
	$\overline{O(n)}$

`select`$(k, A[1..n])$

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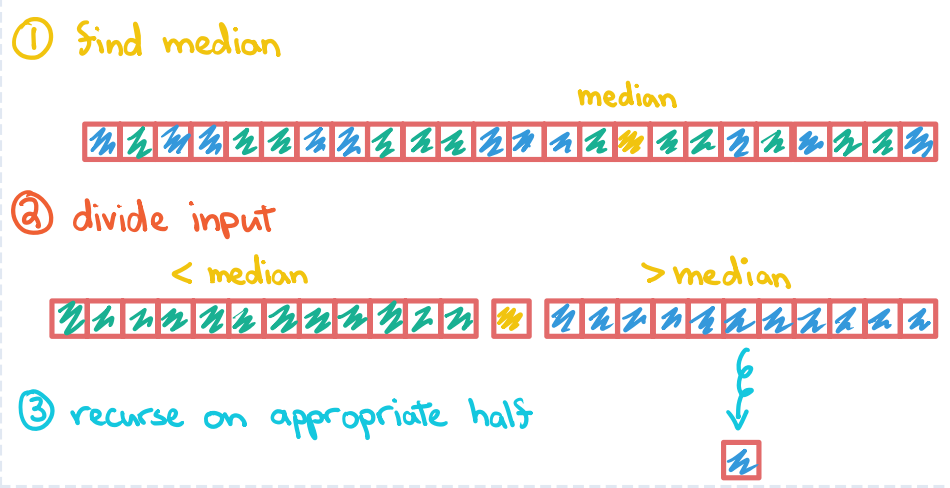

$\Rightarrow O(n)$ time

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Problem: don't have O(n)-time `median`$(A[1..n])$

New thought experiment: suppose O(n)-time subroutine

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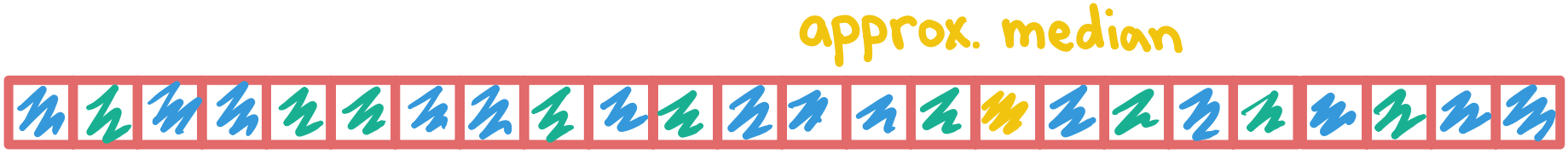
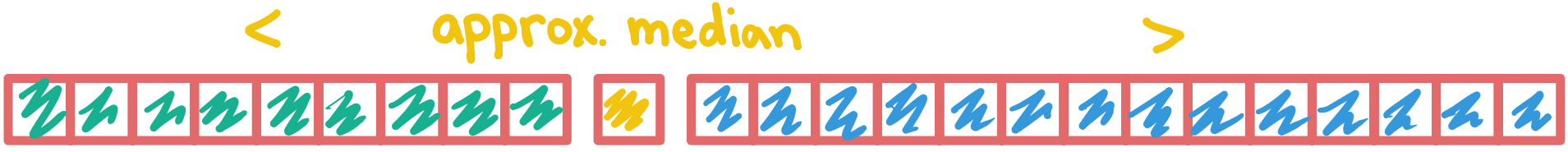
O(n) time

`apx-median`$(A[1..n])$ returns an element $p \in A[1..n]$ with rank between $n/10$ and $9n/10$, in $O(n)$ time.

① find approx. median



② divide input



③ recurse on appropriate half
(somewhat unbalanced)

① Recursive specification

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1. If $n \leq 10$ then find the kth element by brute force.
2. Let $p =$ `apx-median`(A). // *$O(n)$ time by assumption.*
3. Compare p to each element in $A[1..n]$ to identify its rank ℓ. // *$O(n)$.*
4. If $k = \ell$ then return p.
5. If $k < \ell$ then return `select`$(k, A[1..\ell - 1])$, where $B[1..\ell - 1]$ is an array of the $\ell - 1$ elements smaller than p.
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Running time

`select`$(k, A[1..n])$

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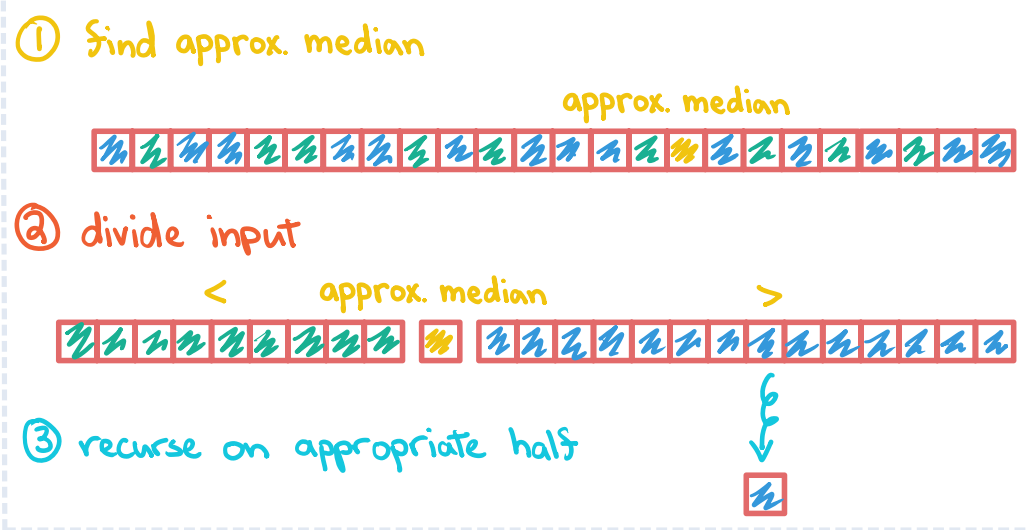
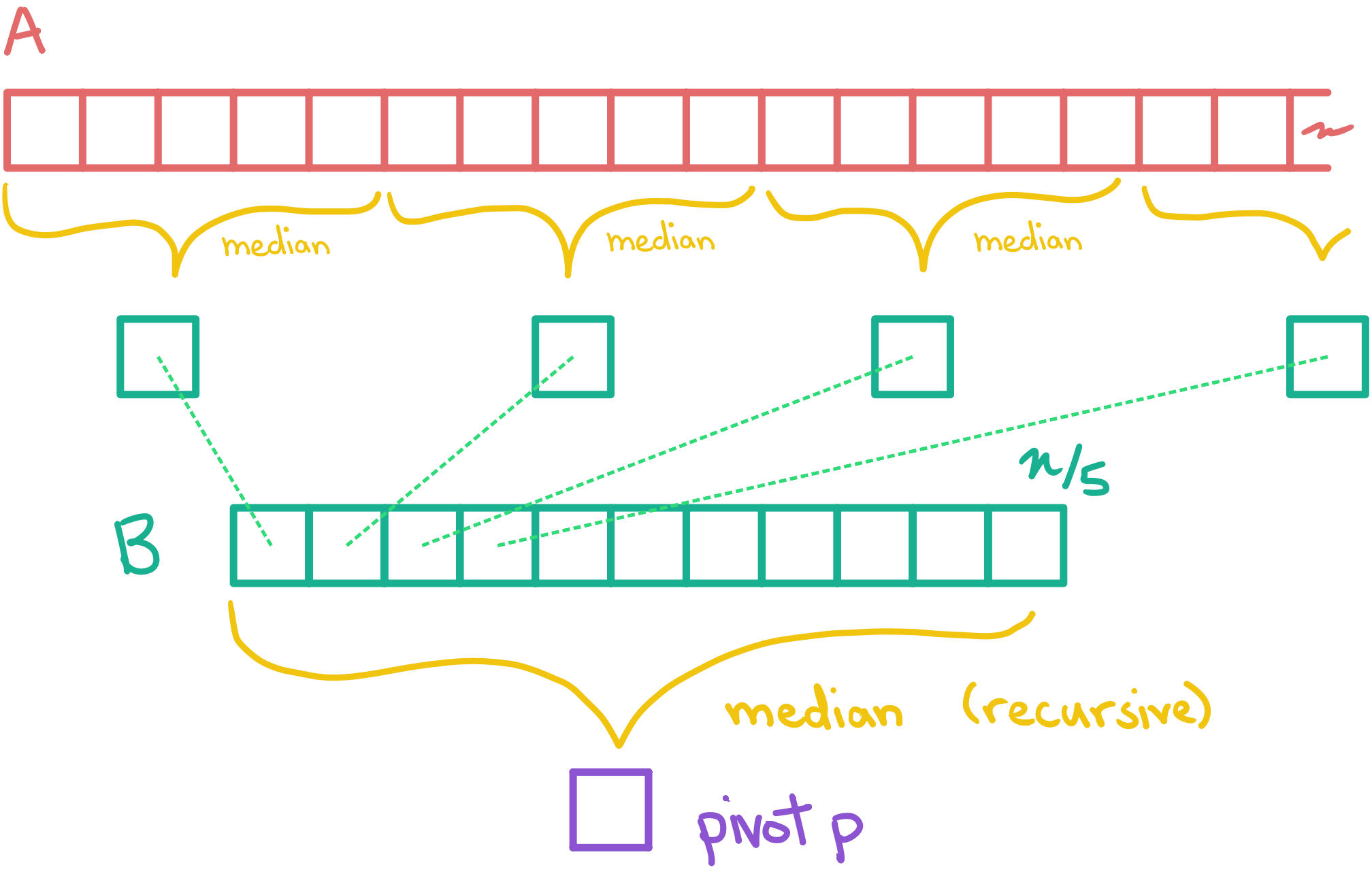


Remains to implement

`apx-median`$(A[1..n])$ returns an element $p \in A[1..n]$ with rank between $n/10$ and $9n/10$, in $O(n)$ time.

"Median of Medians"

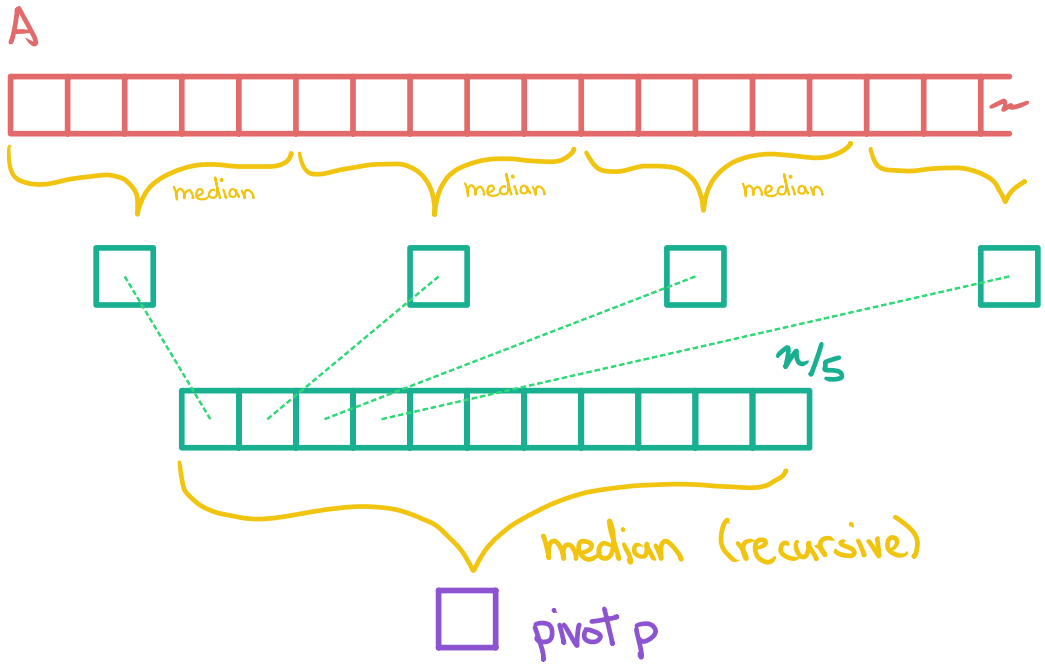
Blum, Floyd, Pratt, Rivest, Tarjan



claim:

M-of-M pivot p has rank

$$\frac{3n}{10} \leq \text{rank}(p) \leq \frac{7n}{10} + 5$$



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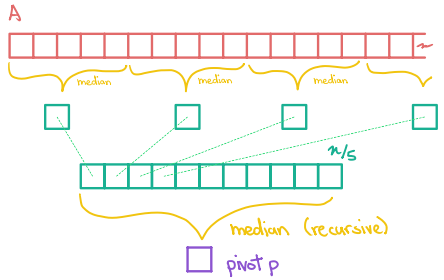

B sorted →

B

n/5

each column sorted ↓

= def. smaller

= def. bigger

$$\frac{n}{5} \times \frac{1}{2} \times 3 = \frac{3n}{10}$$

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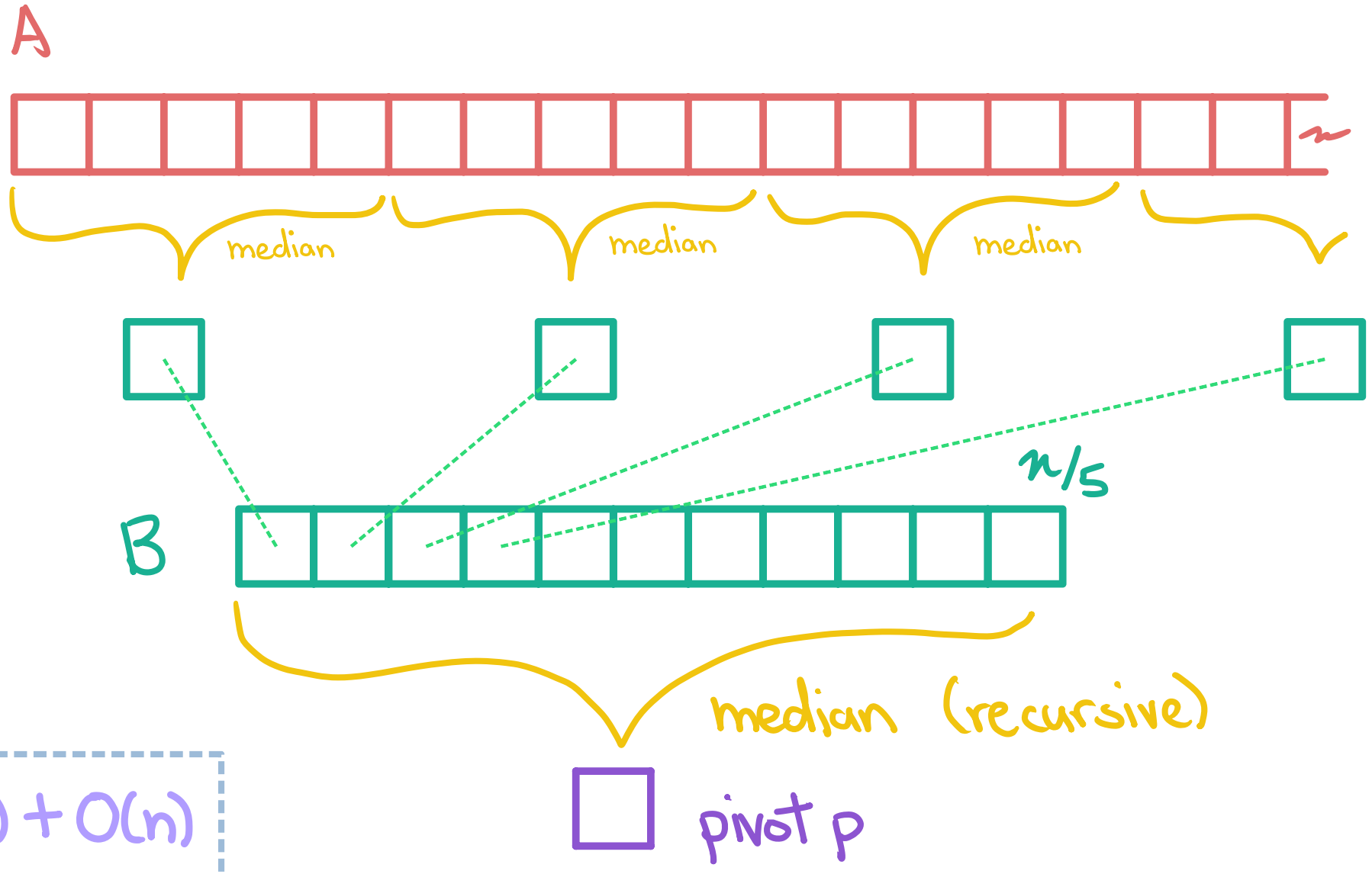
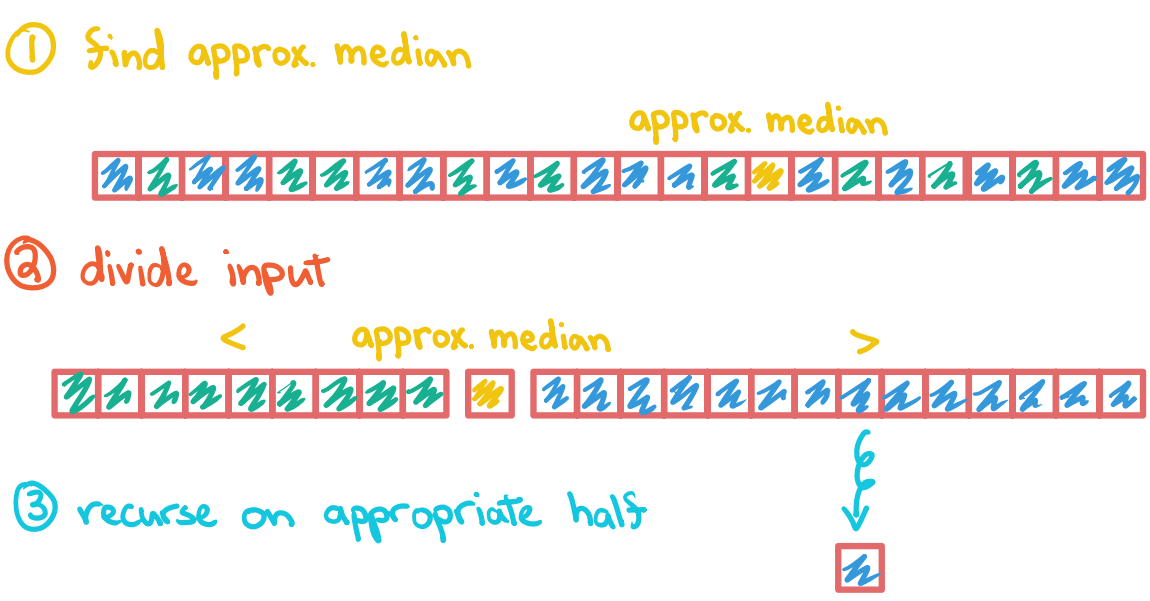
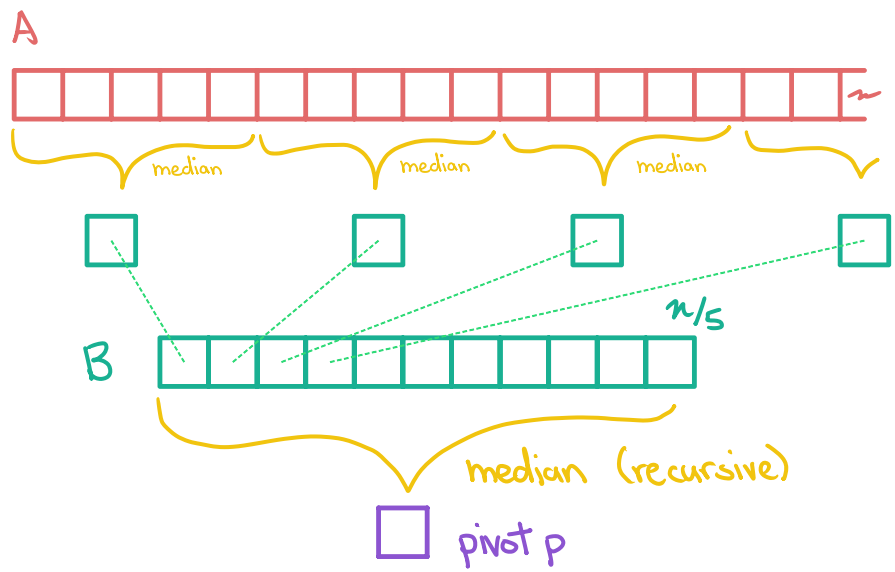


$T(n/5) + O(n)$ time





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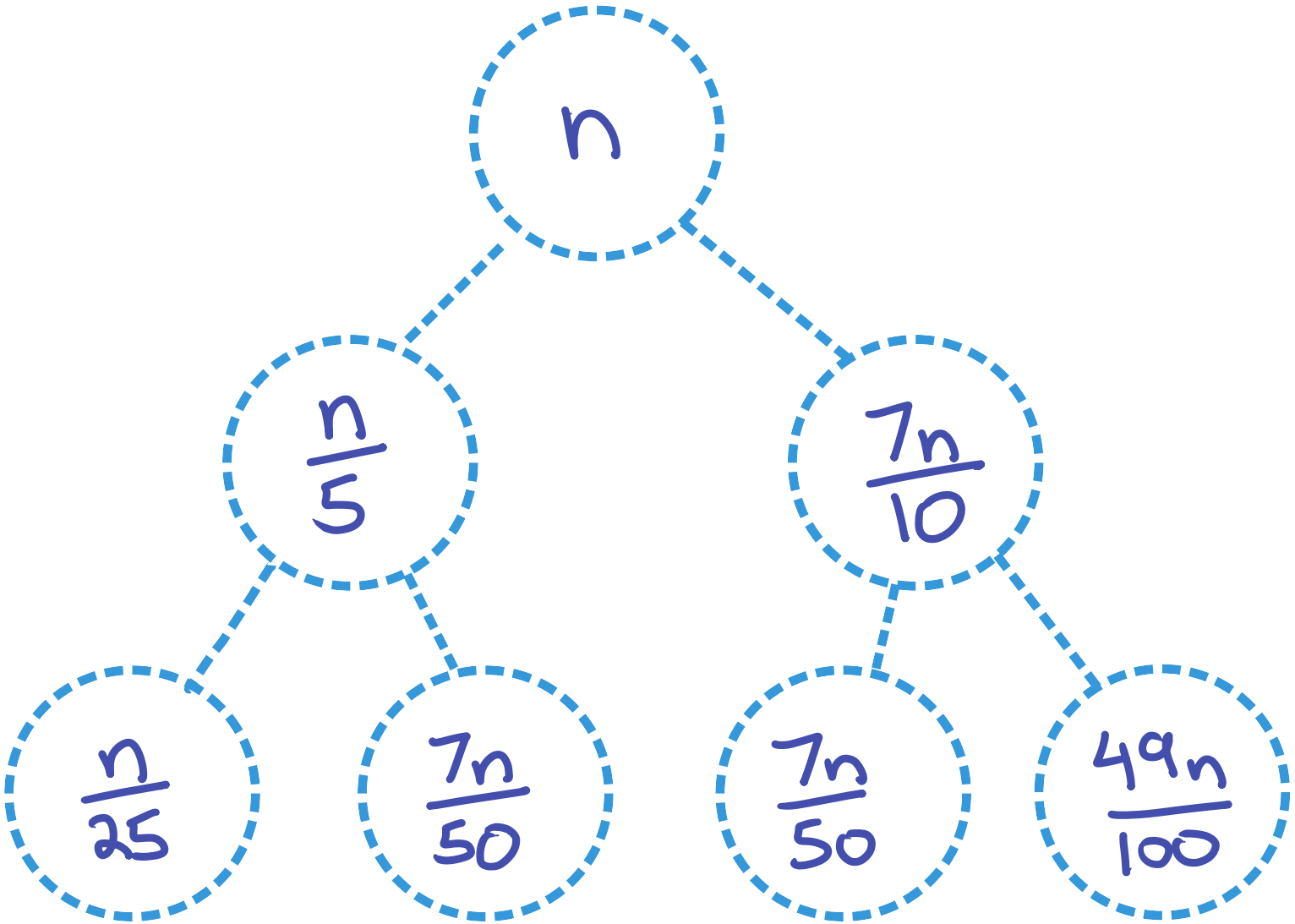
$$T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + O(n)$$



n

$\left(\frac{1}{5}+\frac{7}{10}\right)n$

$\left(\frac{1}{5}+\frac{7}{10}\right)^2 n$

$\left(\frac{1}{5}+\frac{7}{10}\right)^3 n$

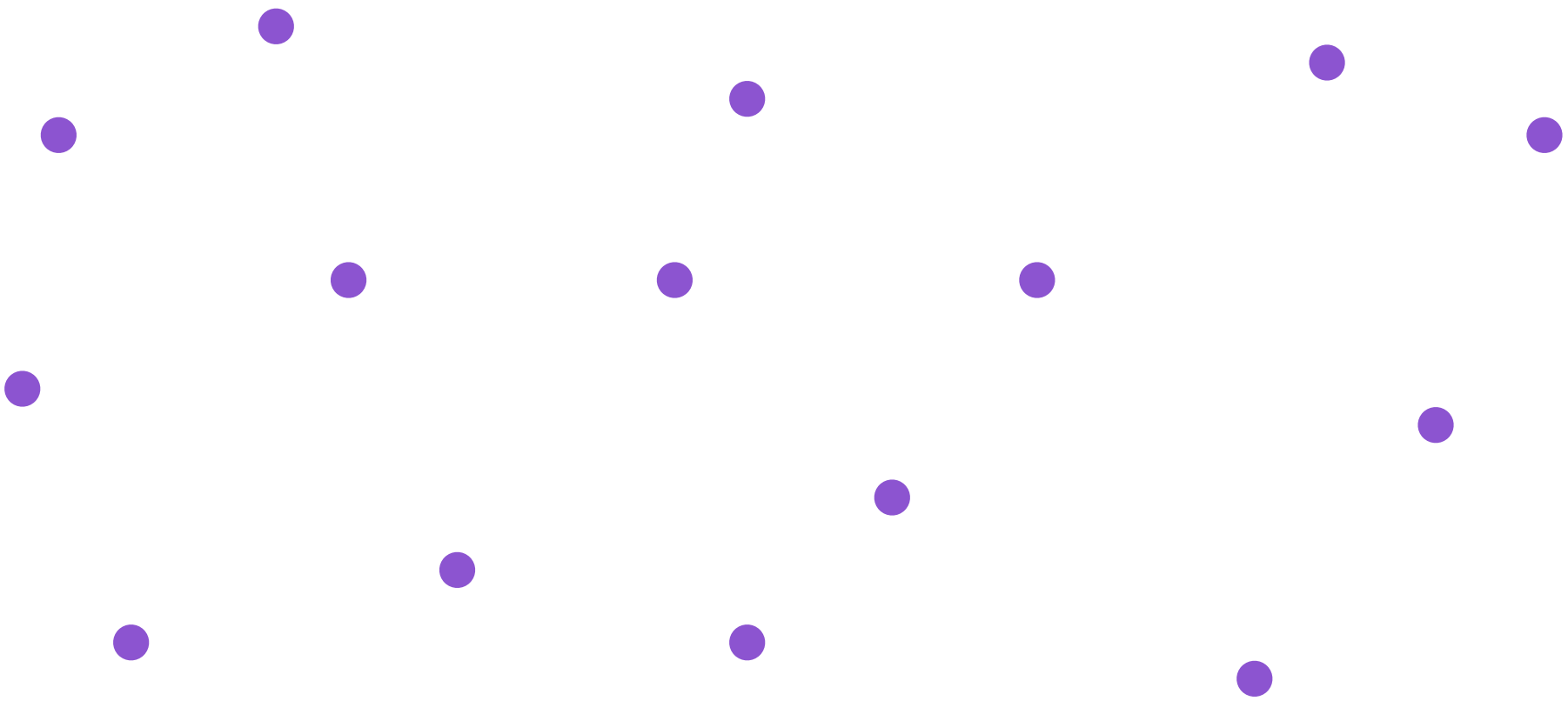
$$\sum_{i=0}^{\infty} \left(\frac{1}{5}+\frac{7}{10}\right)^i n = O(n)$$

Minimum Euclidean distance

Input: n points in $\mathbb{R}^2$

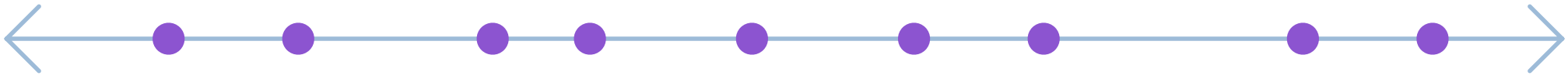
Goal: min distance over all pairs

$\|x-y\| = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$ for $x, y \in \mathbb{R}^2$



$O(n^2)$ obvious. Better?

Warmup: 1 dimension (all y-coordinates = 0)



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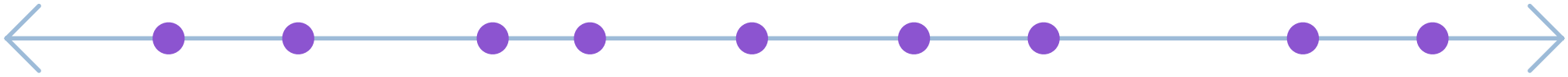

Approach 1: sort and scan. $O(n \log n)$

(hard to generalize to $\mathbb{R}^2$)

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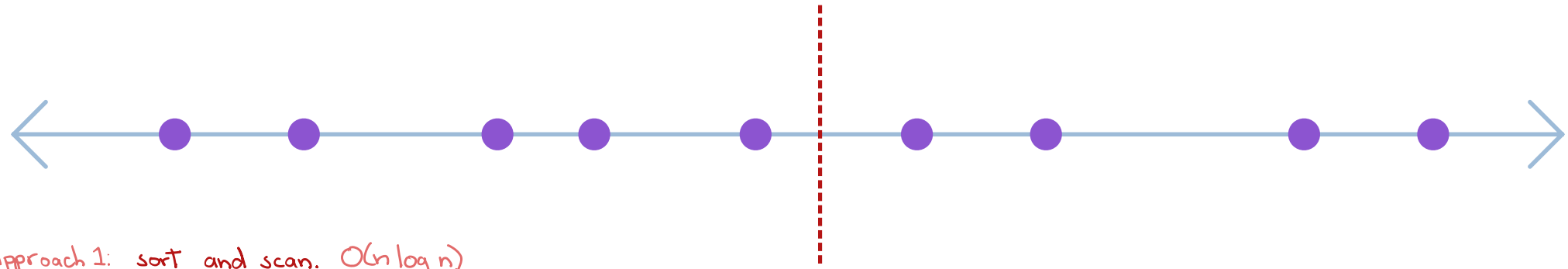

Approach 1: sort and scan. $O(n \log n)$
(hard to generalize to $\mathbb{R}^2$)

Approach 2.

Split at median, recurse

Compare w/ min pair "across" split

$O(n \log n)$ (like merge sort)

seems to generalize better

Generalizing to $\mathbb{R}^2$

1. split at median x-coordinate
2. recurse on each half
   let δ = min distance from the two halves
3. look at "vertical band" of width 2δ around median
4. compute min distance inside band

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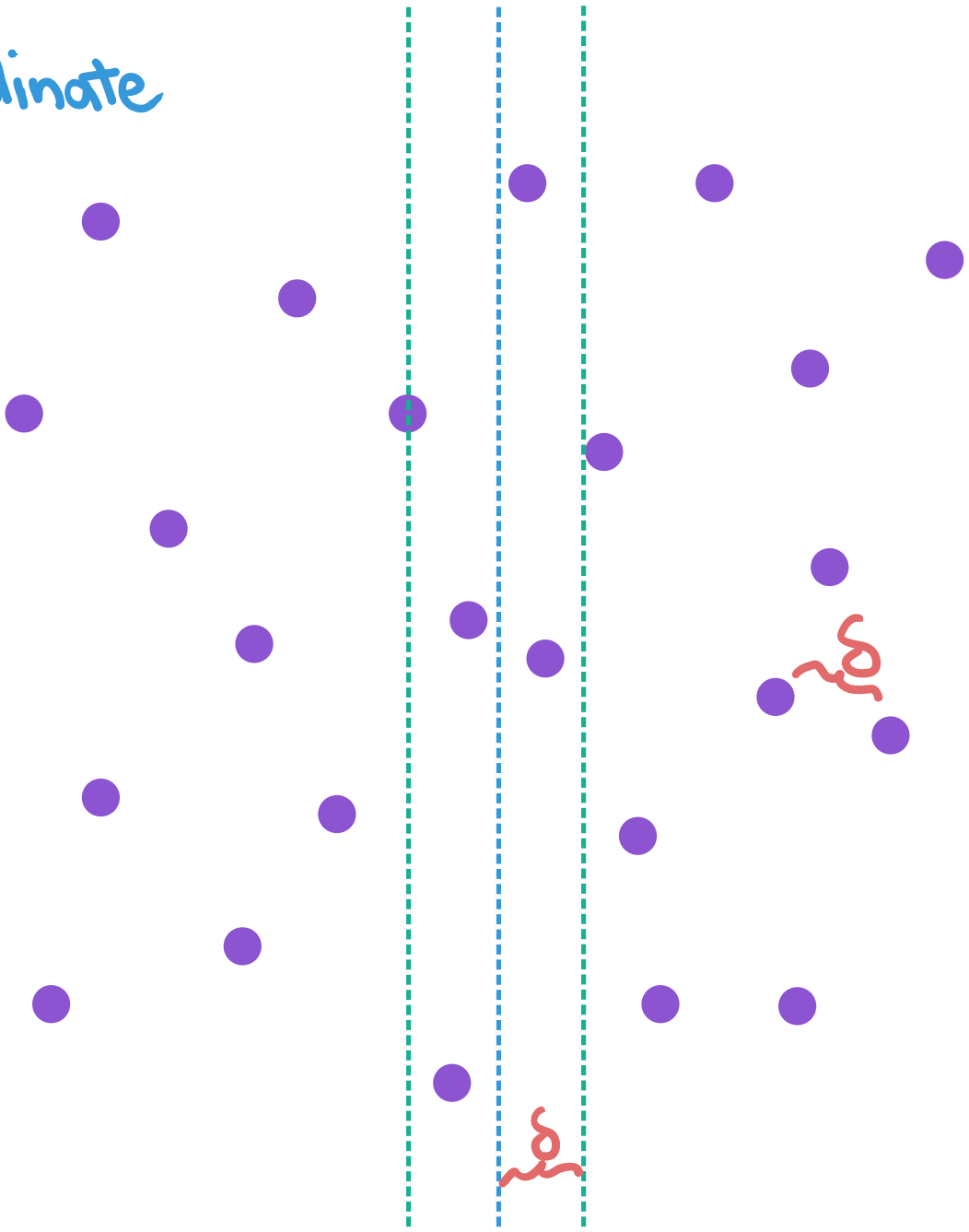
Split at median, recurse

Compare w/ min pair "across" split



1. split at median x-coordinate
2. recurse on each half,
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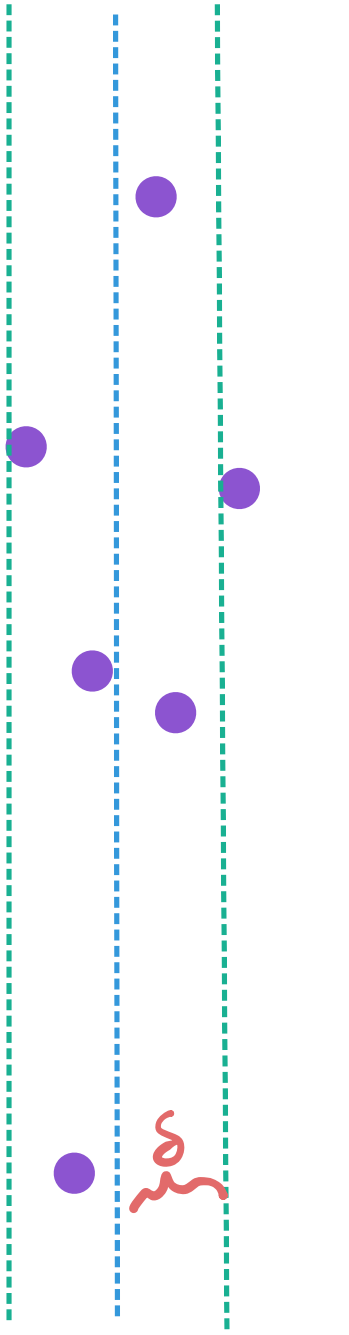

Observations

- very narrow
- points on left side have min distance $\geq \delta$
- points on right side have min distance $\geq \delta$

Intuitively: not dense,
fewer comparisons needed

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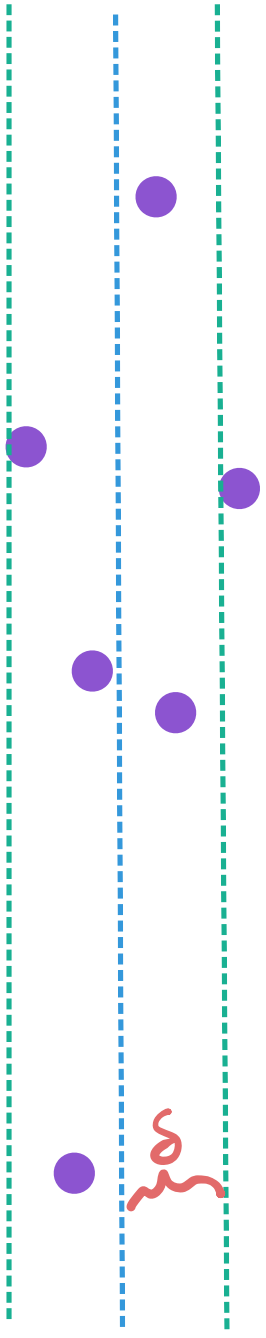

`min-distance(`P`)`

/* *We assume P is sorted by y-coordinate. (Otherwise sort P once in a preprocessing step.)* */

1. If $n \leq 1$ then return $+\infty$.
2. If $n = 2$ then return the distance between the two points in P.
3. Let a be the median of the x-coordinates. // $O(n)$

/* *Divide P along the vertical line $x = a$* */

4. Let $P_1 = \{(x, y) \in P : x < a\}$ and $P_2 = \{(x, y) \in P : x \geq a\}$ // $O(n)$

/* *Find the minimum distances within each half P_1 and P_2 recursively.* */

5. Let $\delta_1 =$ `min-distance(`P_1`)`, $\delta_2 =$ `min-distance(`P_2`)`, and $\delta = \min\{\delta_1, \delta_2\}$. // $2T(n/2)$

/* *It remains to find the minimum distance across P_1 and P_2.* */

6. Let $P_3 = \{(x, y) \in P : |x - a| < \delta\}$ and let $p_1, \ldots, p_k$ list P_3 in decreasing order of y-coordinate. // $O(n)$
7. Let δ_3 be the minimum of $\|p_i - p_j\|$ over all i, j with $i < j < i + 10$. // $O(n)$
8. Return $\min\{\delta, \delta_3\}$.

(a = median x-coordinate)

Lemma 11.3. *Let $b \in \mathbb{R}$. There are at most 10 points in P with coordinate in the rectangle $[a-\delta, a+\delta] \times [b-\delta, b]$,*

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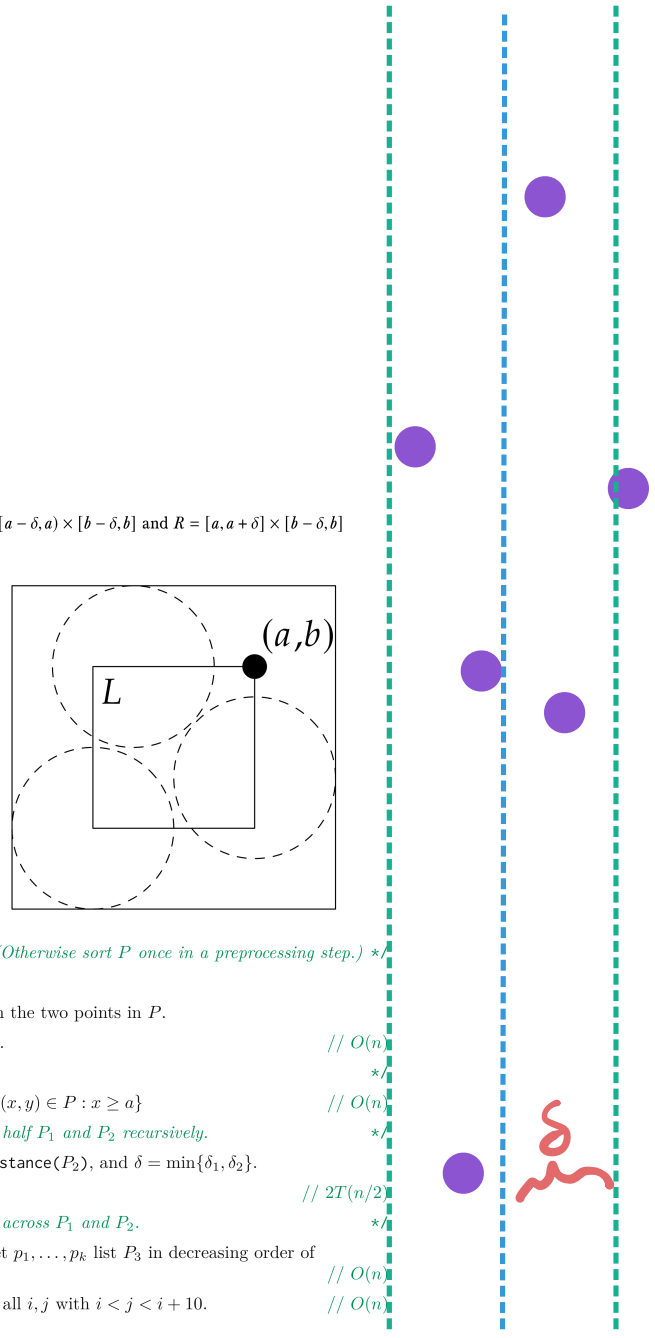

$L = [a-\delta, a) \times [b-\delta, b]$ and $R = [a, a+\delta] \times [b-\delta, b]$

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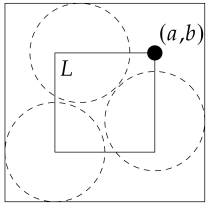

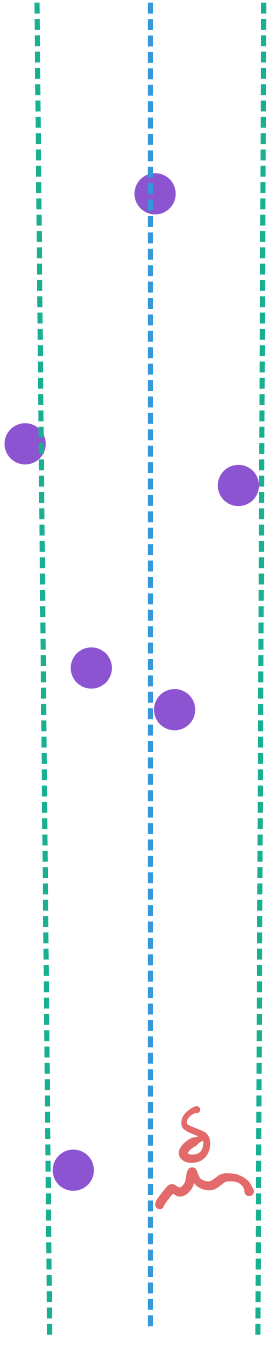

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4. Let $P_1 = \{(x,y) \in P : x < a\}$ and $P_2 = \{(x,y) \in P : x \geq a\}$ // $O(n)$

/ Find the minimum distances within each half P_1 and P_2 recursively. */*

5. Let $\delta_1 =$ `min-distance(`P_1`)`, $\delta_2 =$ `min-distance(`P_2`)`, and $\delta = \min\{\delta_1, \delta_2\}$. // $2T(n/2)$

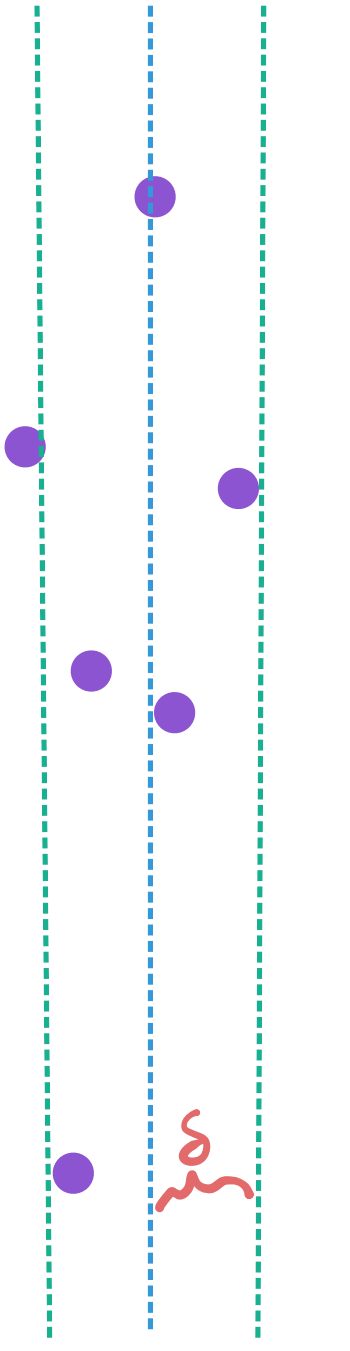
/ It remains to find the minimum distance across P_1 and P_2. */*

6. Let $P_3 = \{(x,y) \in P : |x - a| < \delta\}$ and let $p_1, \ldots, p_k$ list P_3 in decreasing order of y-coordinate. // $O(n)$
7. Let δ_3 be the minimum of $\|p_i - p_j\|$ over all i, j with $i < j < i + 10$. // $O(n)$
8. Return $\min\{\delta, \delta_3\}$.

Running time

Observations
- very narrow
- points on left side have min distance $\geq \delta$
- points on right side have min distance $\geq \delta$

Intuitively: not dense, fewer comparisons needed



`min-distance(`P`)`

/ We assume P is sorted by y-coordinate. (Otherwise sort P once in a preprocessing step.) */*

1. If $n \leq 1$ then return $+\infty$.
2. If $n = 2$ then return the distance between the two points in P.
3. Let a be the median of the x-coordinates. // $O(n)$

/ Divide P along the vertical line $x = a$ */*

4. Let $P_1 = \{(x, y) \in P : x < a\}$ and $P_2 = \{(x, y) \in P : x \geq a\}$ // $O(n)$

/ Find the minimum distances within each half P_1 and P_2 recursively. */*

5. Let $\delta_1 =$ `min-distance(`P_1`)`, $\delta_2 =$ `min-distance(`P_2`)`, and $\delta = \min\{\delta_1, \delta_2\}$. // $2T(n/2)$

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7. Let δ_3 be the minimum of $\|p_i - p_j\|$ over all i, j with $i < j < i + 10$. // $O(n)$
8. Return $\min\{\delta, \delta_3\}$.

$T(n) = 2T(n/2) + O(n) = O(n \log n)$

Multiplying n-bit integers

Karatsuba

Input: $x, y \in \{0,1\}^n$

Output: product $(xy) \in \{0,1\}^{2n}$

$$
\begin{array}{r}
1234 \\
\times\ 5678 \\
\hline
9872 \\
8638 \\
+\ \ 6 \\
\hline
\end{array}
$$

$O(n^2)$ time

Better?