

Searching & Sorting Graphs

Today: directed graphs

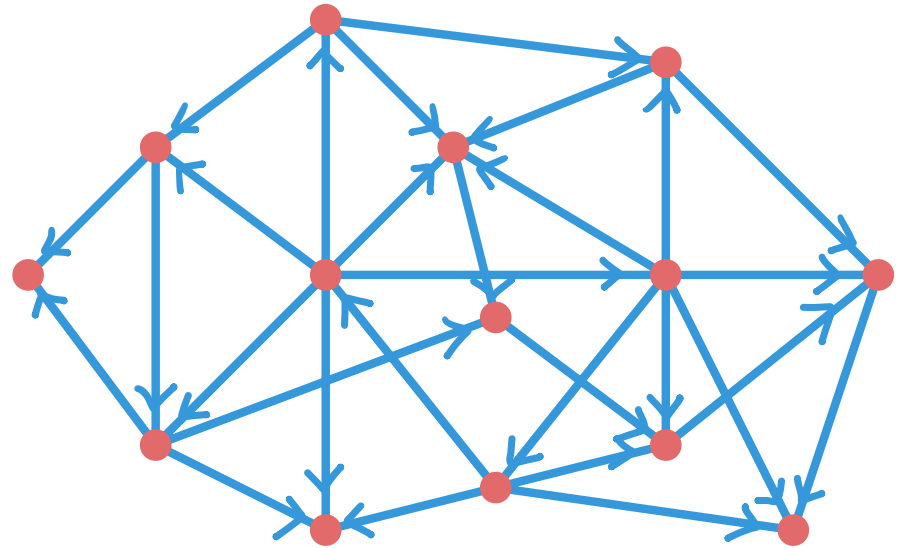
acyclic graphs,

topological sort,

cycles, reachability,

depth-first search,

strongly connected components



(which give a lot of structure to a very simple and broadly applicable model)

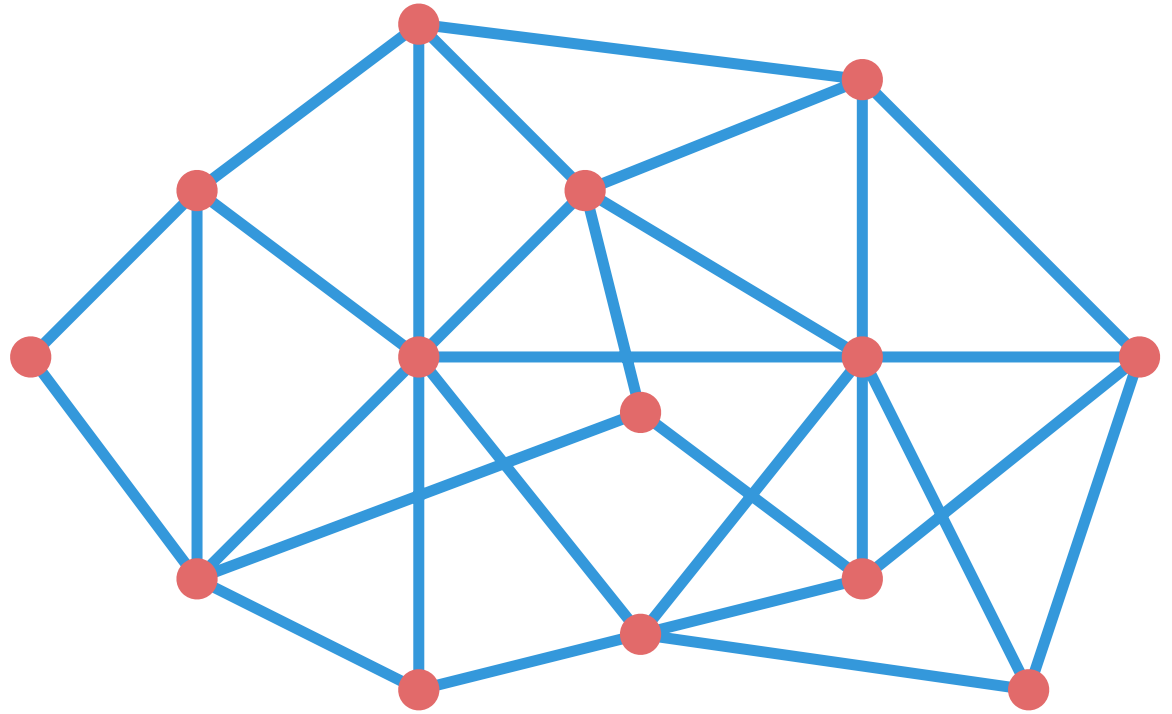
Graphs

$$G=(V, E)$$

$$n=|V|, m=|E|$$

vertices •

edges —————

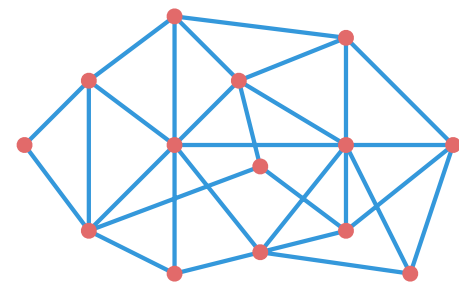


undirected
edge
between u and v



directed
edge
from u to v

Representing graphs



① "Adjacency list"

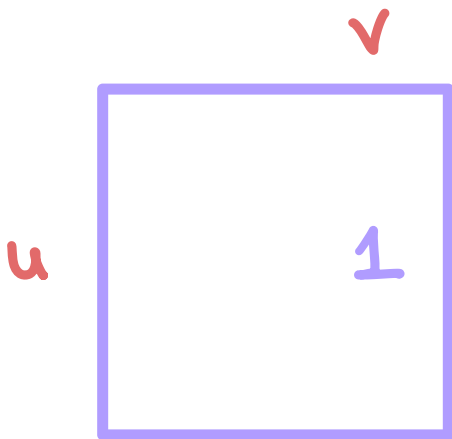


for $v \in V$, list all (incoming, outgoing) edges of v

e.g. $v.out = [a, b, c, \dots]$, $v.in = [d, e, f, \dots]$

② "Adjacency matrix"

$A \in \mathbb{R}^{V \times V}$ w/ $A(u, v) = \text{weight of edge from } u \text{ to } v$
(=0 if no edge)



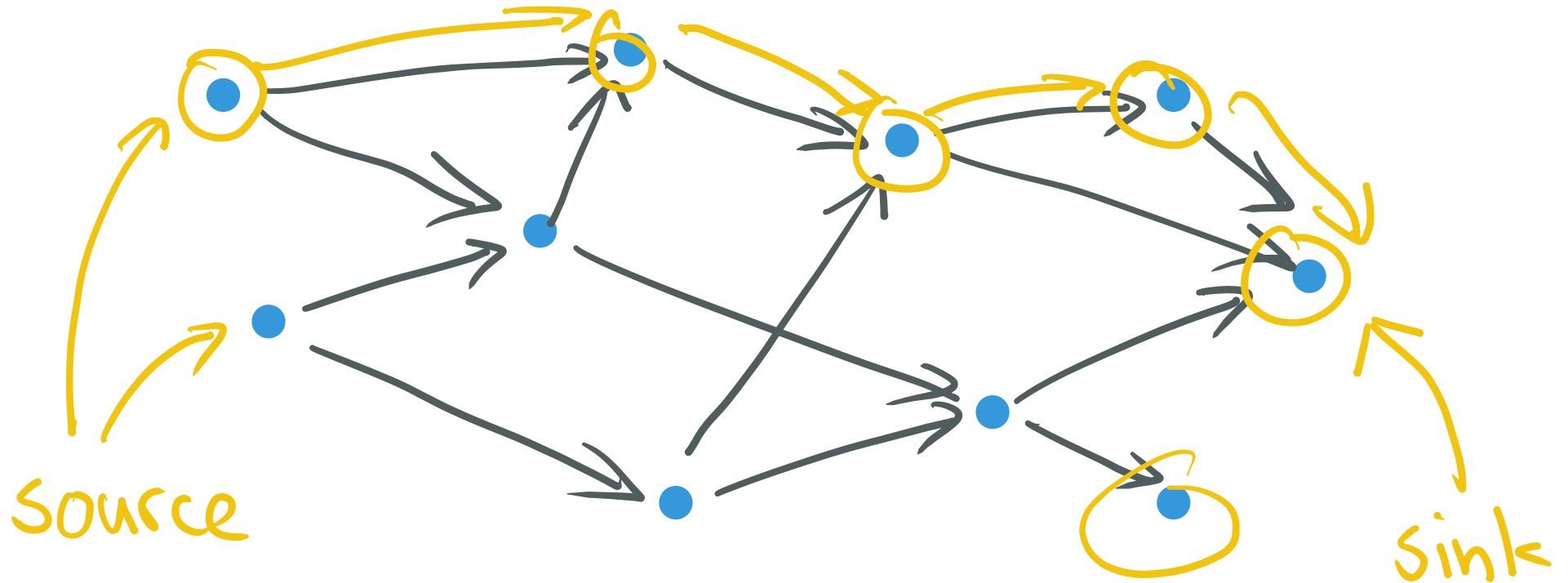
I. DAG's and topological orderings

II. Searching graphs

III. Sorting graphs

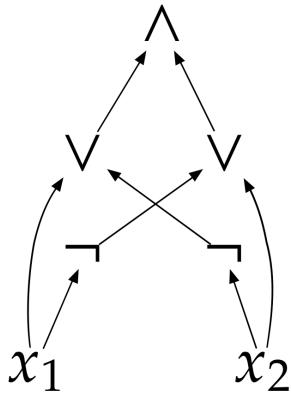
Directed Acyclic Graphs (DAG's)

(no cycles)

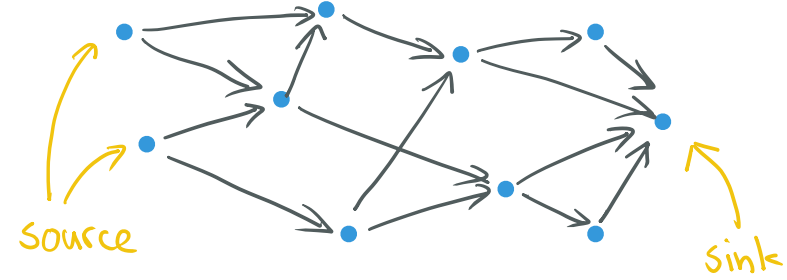


e.g. circuits

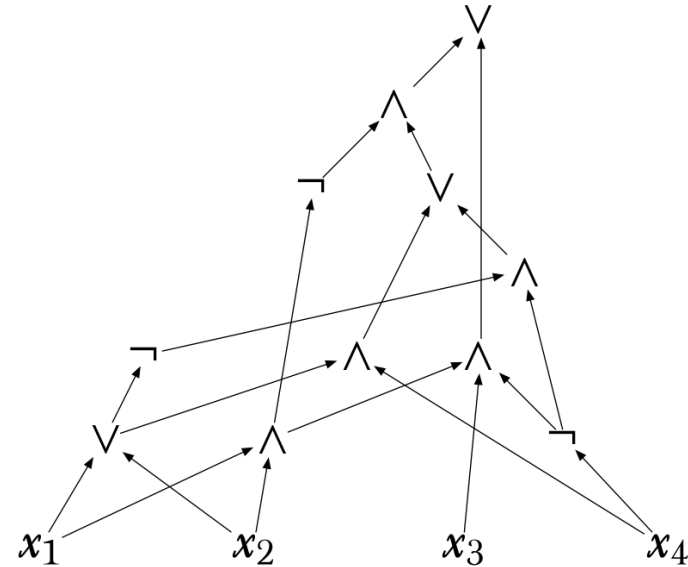
XOR ("exclusive-or")
(TRUE iff $x_1 \neq x_2$)



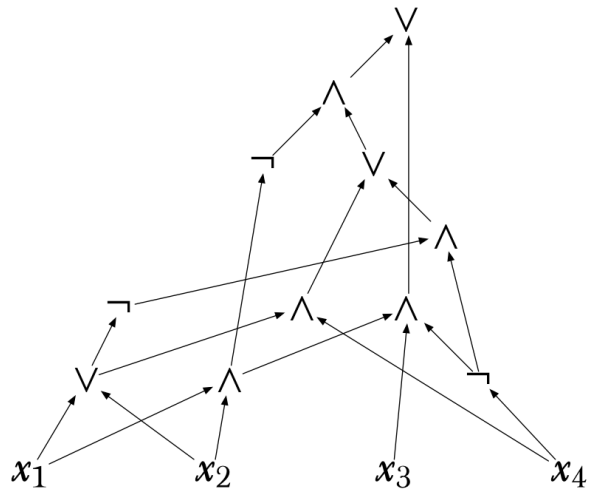
$$f(x_1, x_2) = (x_1 \wedge \bar{x}_2) \vee (\bar{x}_1 \wedge x_2)$$



Addition



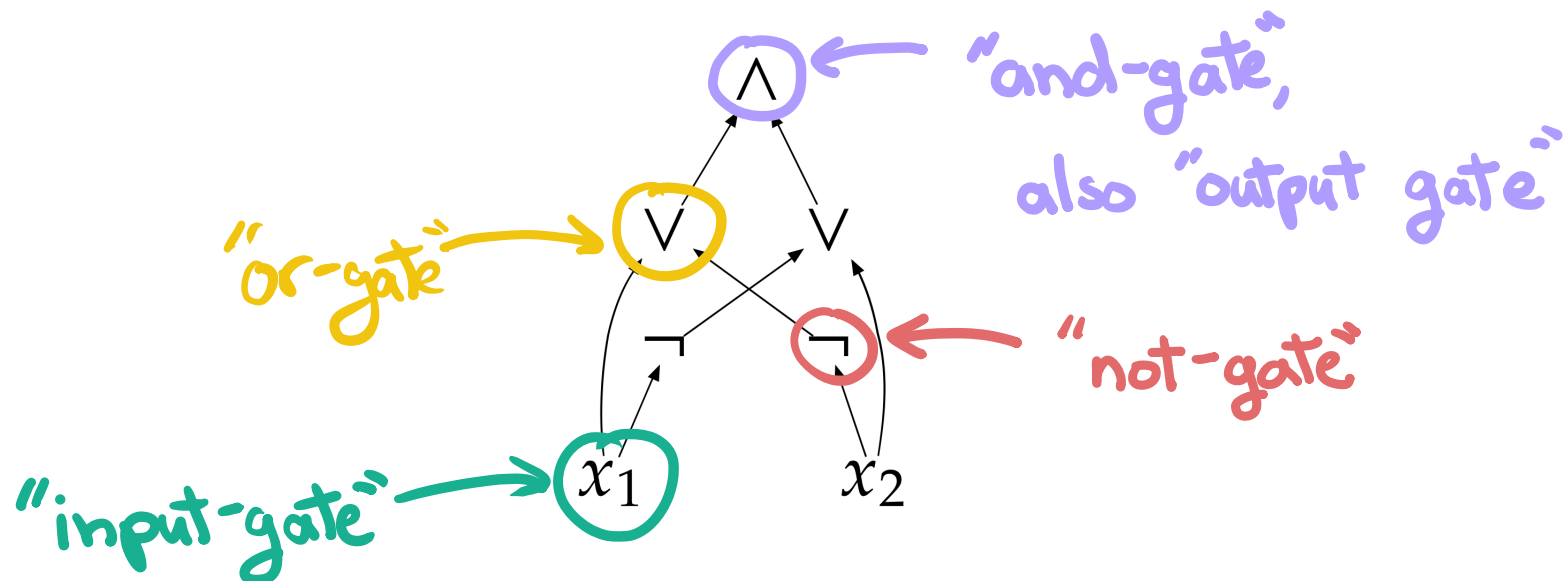
$$f(x_1, x_2, x_3, x_4) = \begin{cases} 1 & \text{if } x_1 + x_2 = x_3 x_4 \\ 0 & \text{otherwise} \end{cases}$$



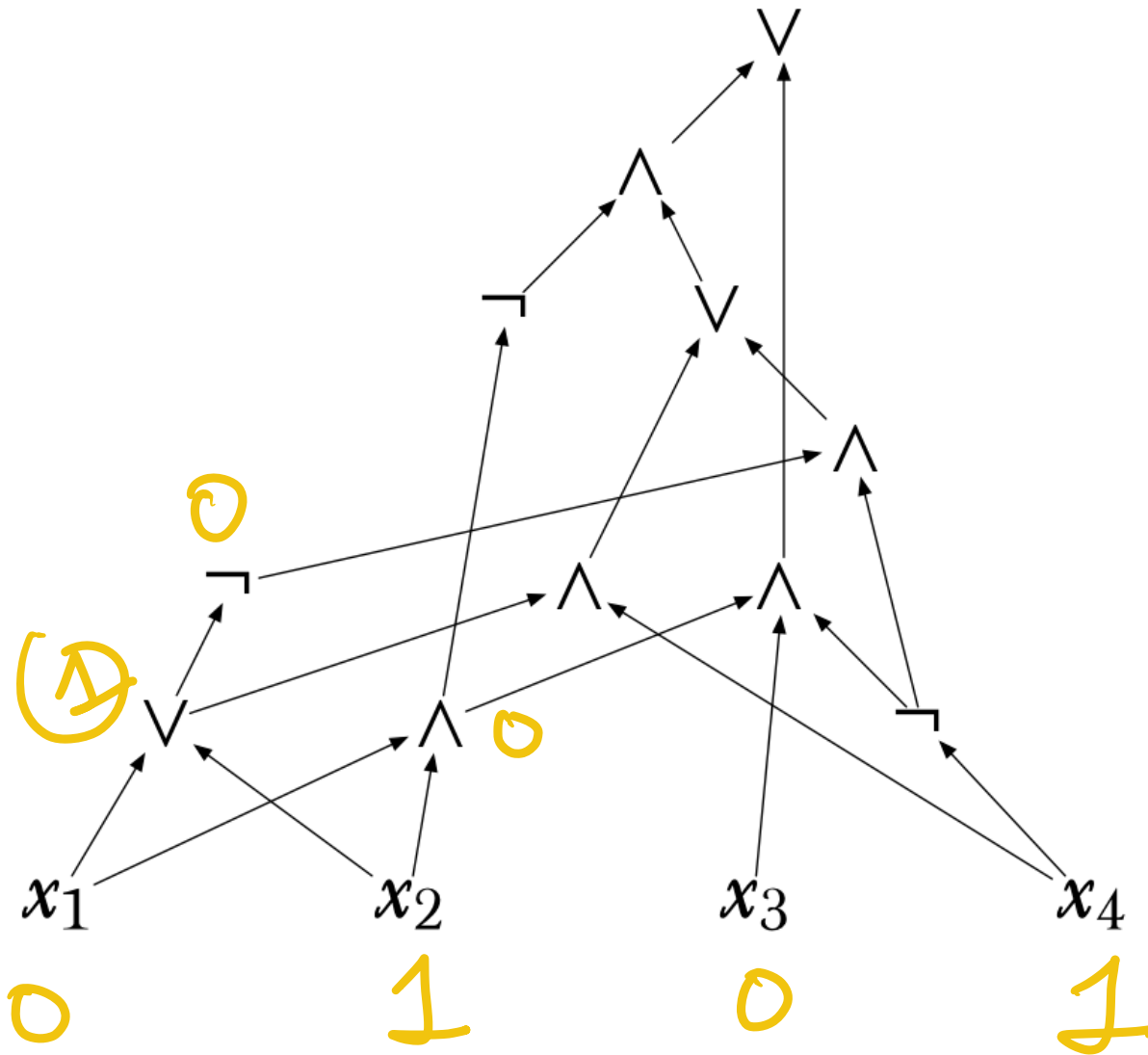
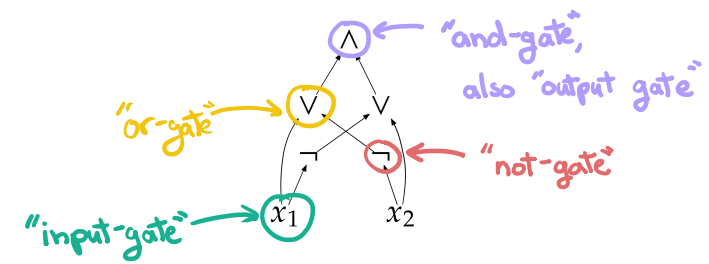
(definition)

logical gates arranged as vertices
in a directed acyclic graph

"gates": take some inputs bits, output 1 bit



How to evaluate a circuit?



(conceptually)

"bottom up,
one at a time"

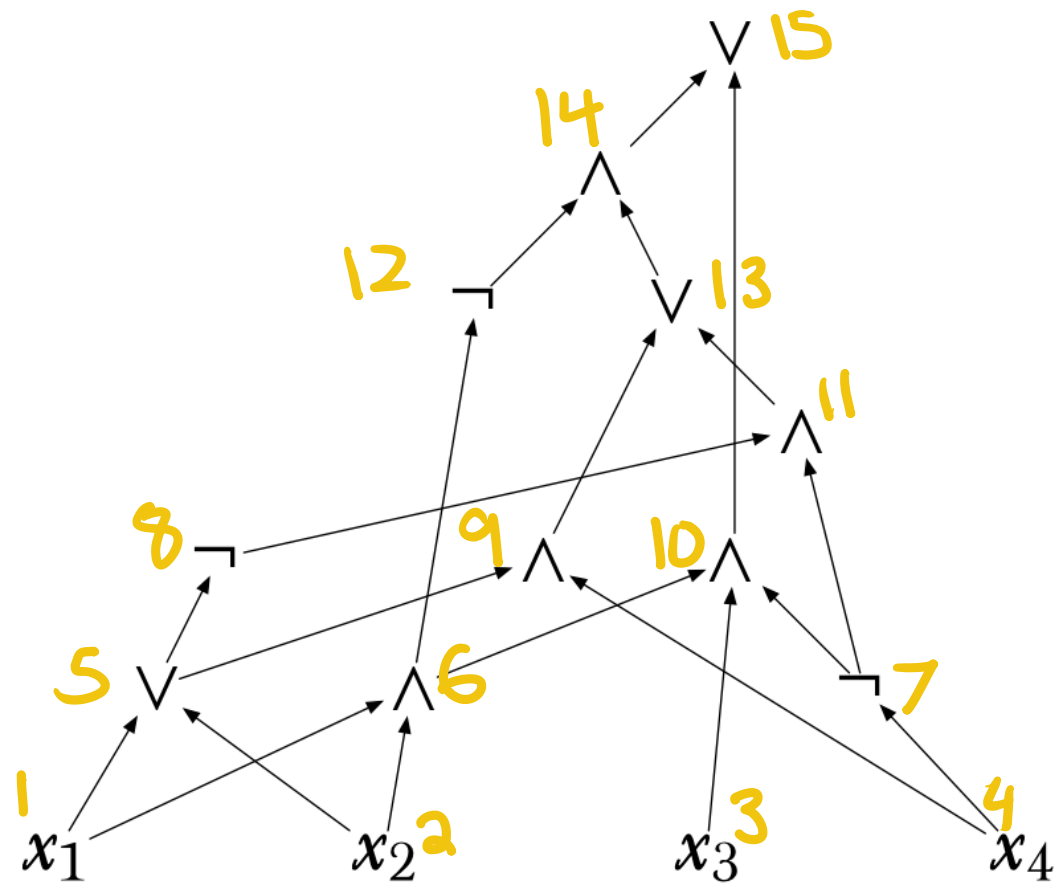
Topological order

think of arcs as " $\leq$ "

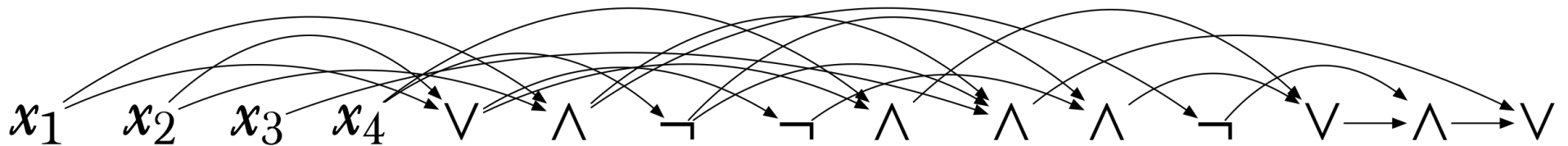
(directed edge)

$u \leq v$

this defines a partial order on V

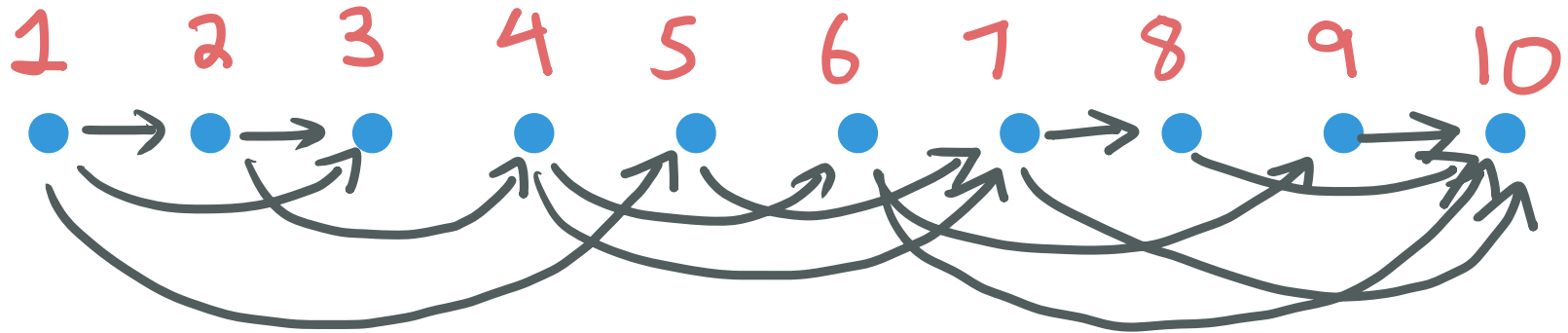
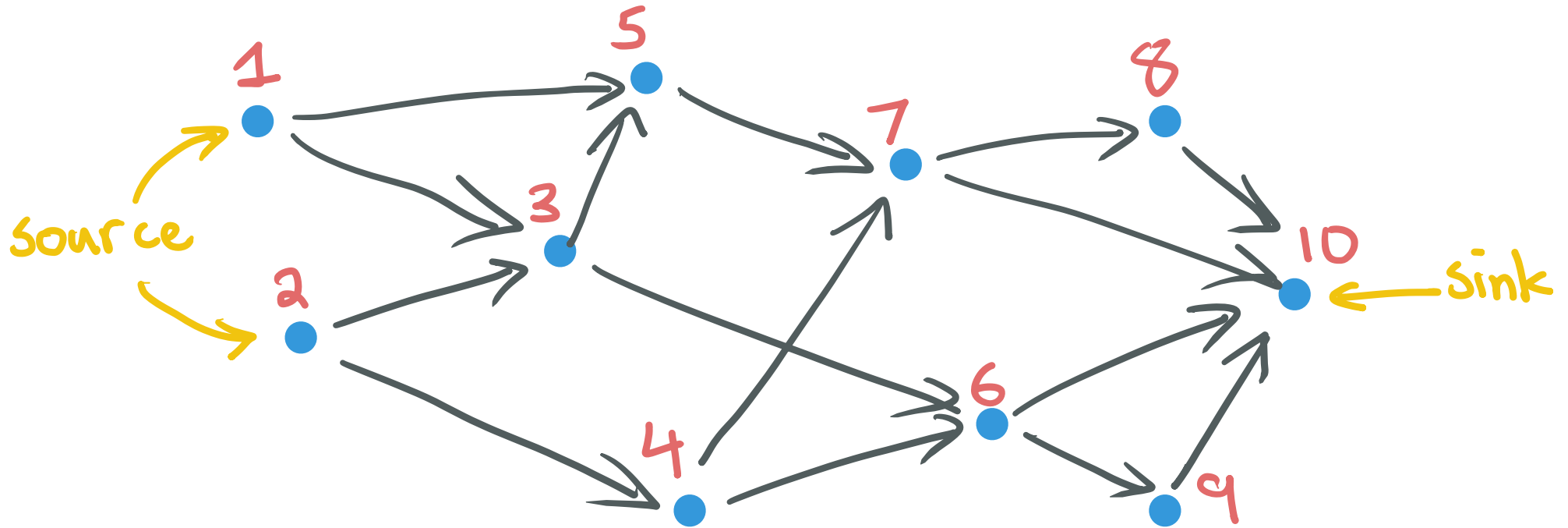
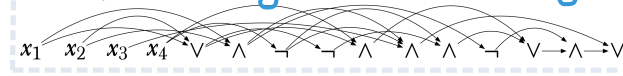


Topological ordering (of V):
any ordering consistent w/ edges



How to topologically sort:
repeatedly remove sources

Topological ordering (of V):
any ordering consistent w/ edges

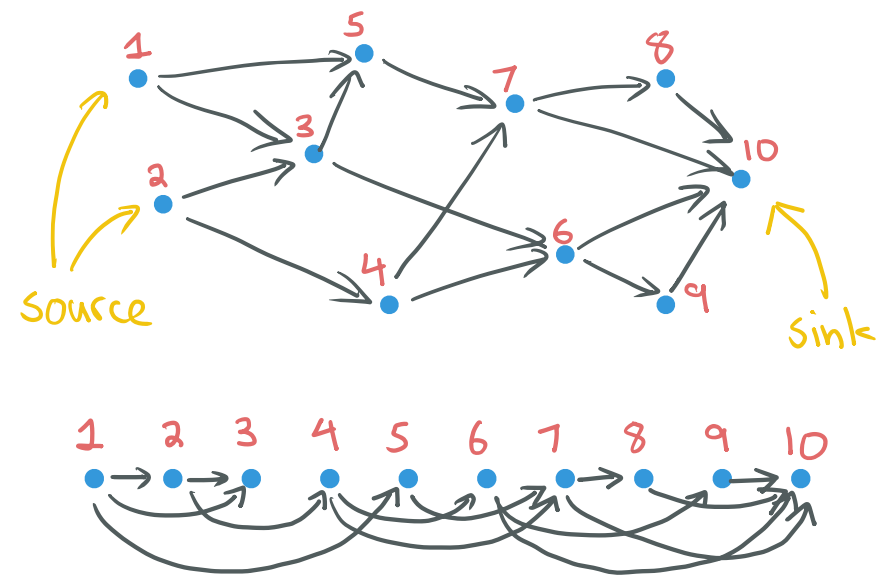


source-peeling($G = (V, E)$)

/ We assume G is a DAG.*

1. While there is a source $v \in V$
 - A. Append v to the output list.
 - B. Remove v and all incident edges from G .

↑ easy to make linear time by keeping track of in-degrees

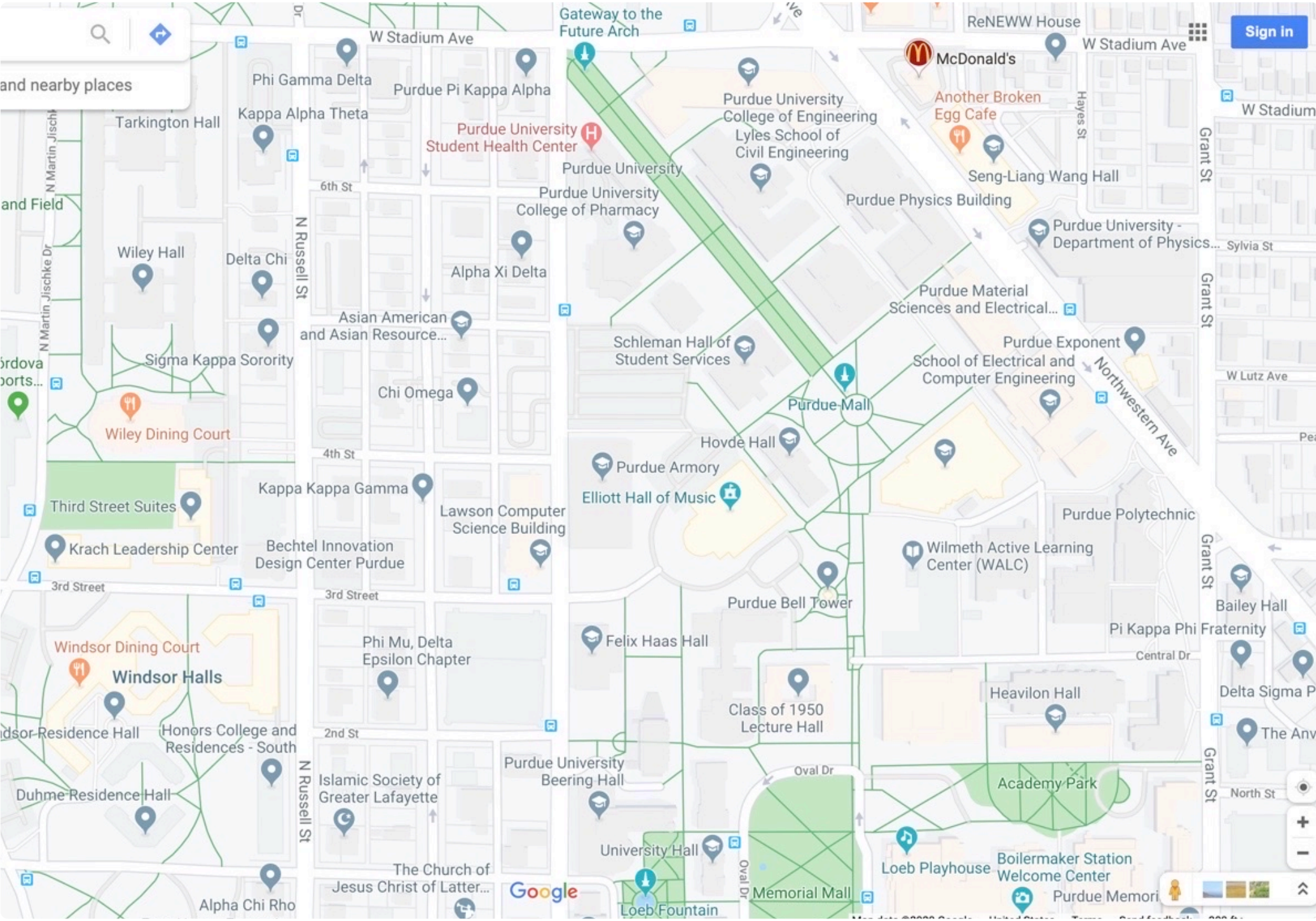


Theorem 6.1. *Given a DAG G with m edges and n vertices, one can compute a topological ordering of V in $O(m + n)$ time.*

I. DAG's and topological orderings

II. Searching graphs

III. Sorting graphs



and nearby places

Sign in

Google

CAT MAZE

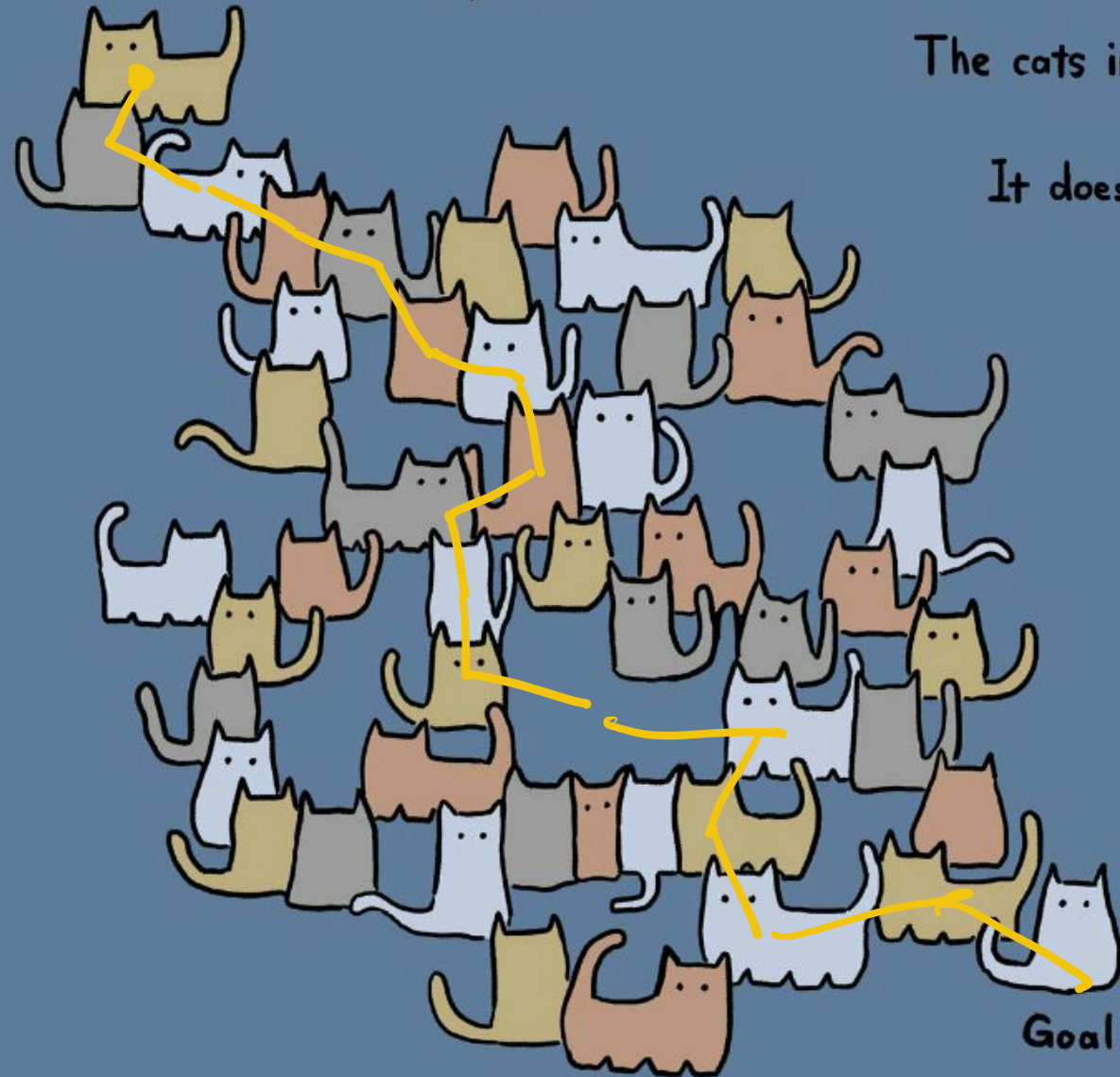
by Tatsunami

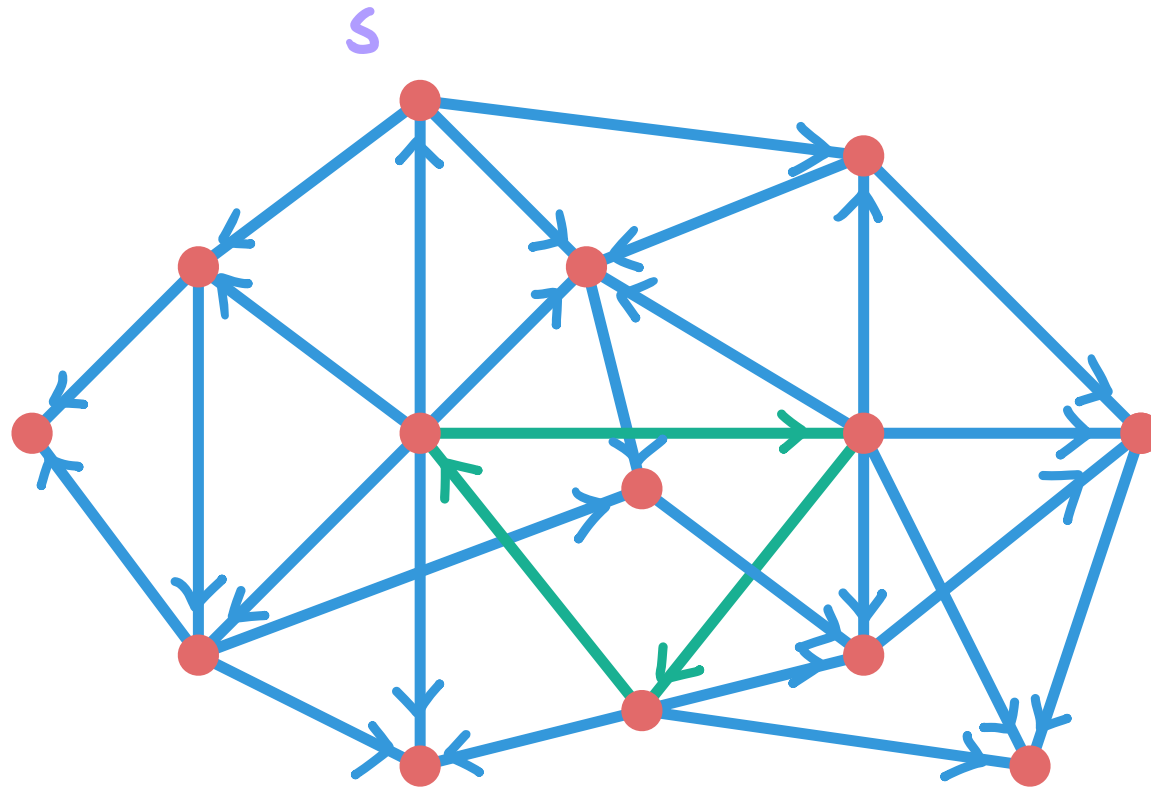
Start

Reach the Goal by following the cats, alternating between cats facing the front and cats facing backwards.

The cats in your path must be connected or overlapping.

It doesn't matter what color or size they are.





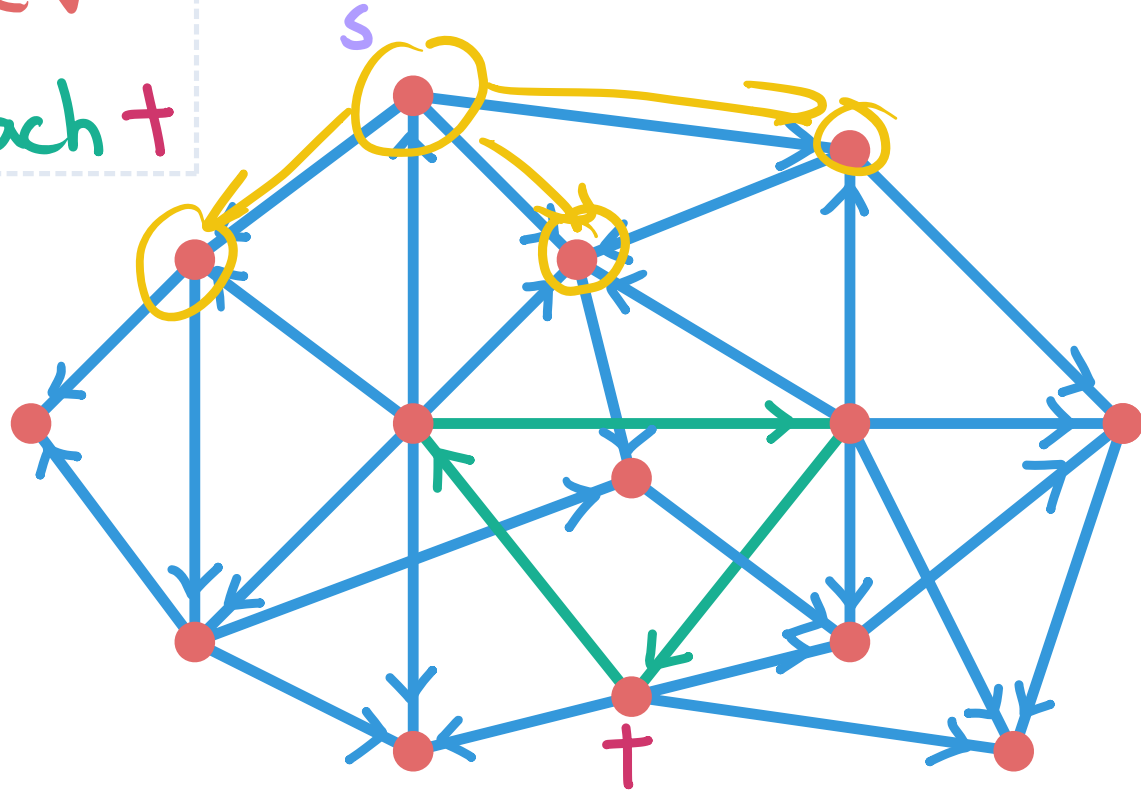
can s reach all the vertices? (no)

Goal: design an algorithm to figure this out automatically

Input: $G=(V,E)$, $s, t \in V$

Goal: decide if s can reach t

$\text{recursive-search}(s,t) =$
TRUE if s can reach t ,
and FALSE otherwise



$\text{Recursive-search}(s,t)$

1. if $s=t$ then return TRUE

2. for each edge (s,v)

(a) if $\text{recursive-search}(v,t) = \text{TRUE}$

then return TRUE

3. return FALSE

$\text{recursive-search}(s,t) = \text{TRUE}$ if s can reach t ,
and FALSE otherwise

$\text{recursive-search}(s,t)$

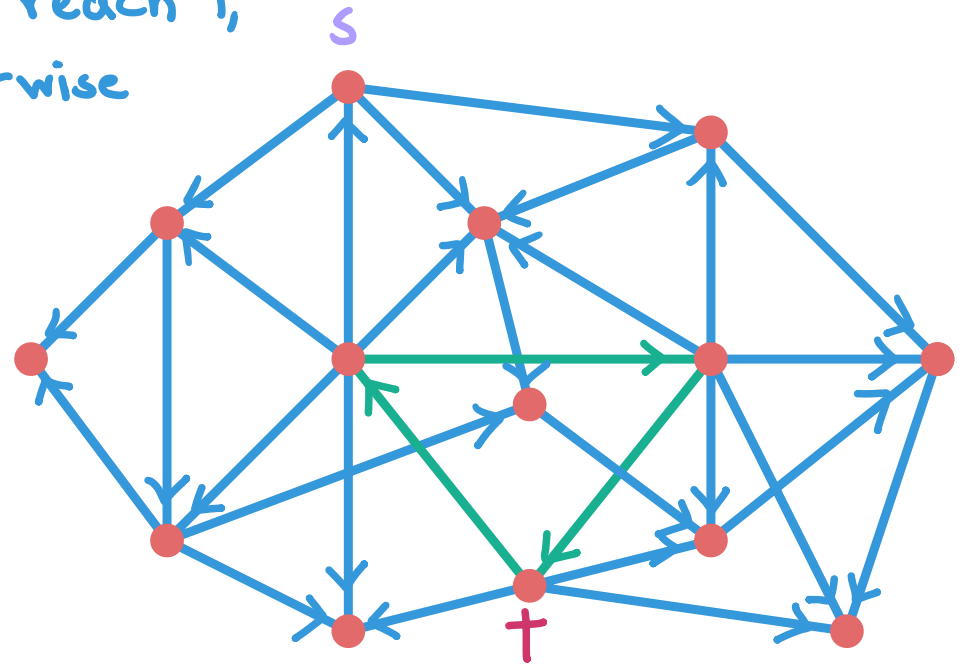
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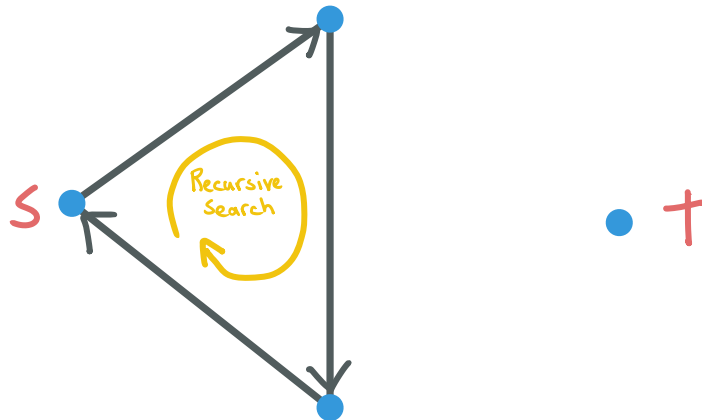
3. return FALSE



Argument: if s can reach t , and $s \neq t$, then there must
be an edge (s,v) s.t. v can reach t .

recursively, $\text{recursive-search}(v,t)$ returns TRUE .

makes sense.
but what about



what went wrong?

Argument: if s can reach t , and $s \neq v$, then there must be an edge (s, v) s.t. v can reach t .

recursively, $\text{recursive-search}(v, t)$ returns TRUE.

recursion requires
induction to be correct

how to fix this?

$\text{recursive-search}(s, t) = \text{TRUE}$ if s can reach t ,
and FALSE otherwise

$\text{recursive-search}(s, t)$

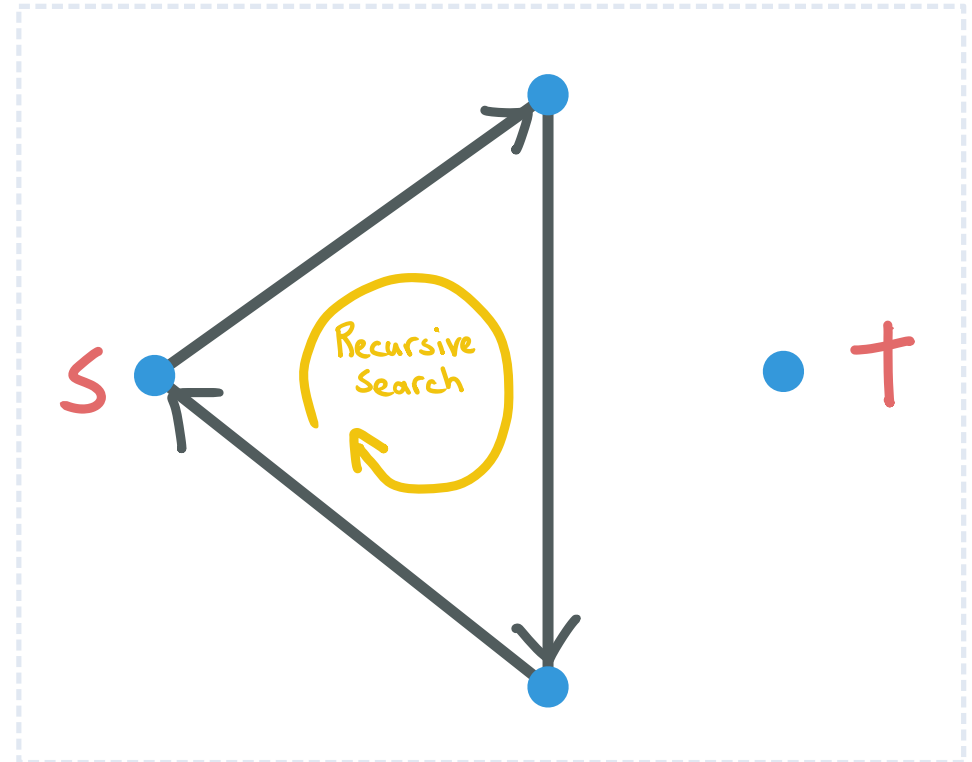
1. if $s = t$ then return TRUE

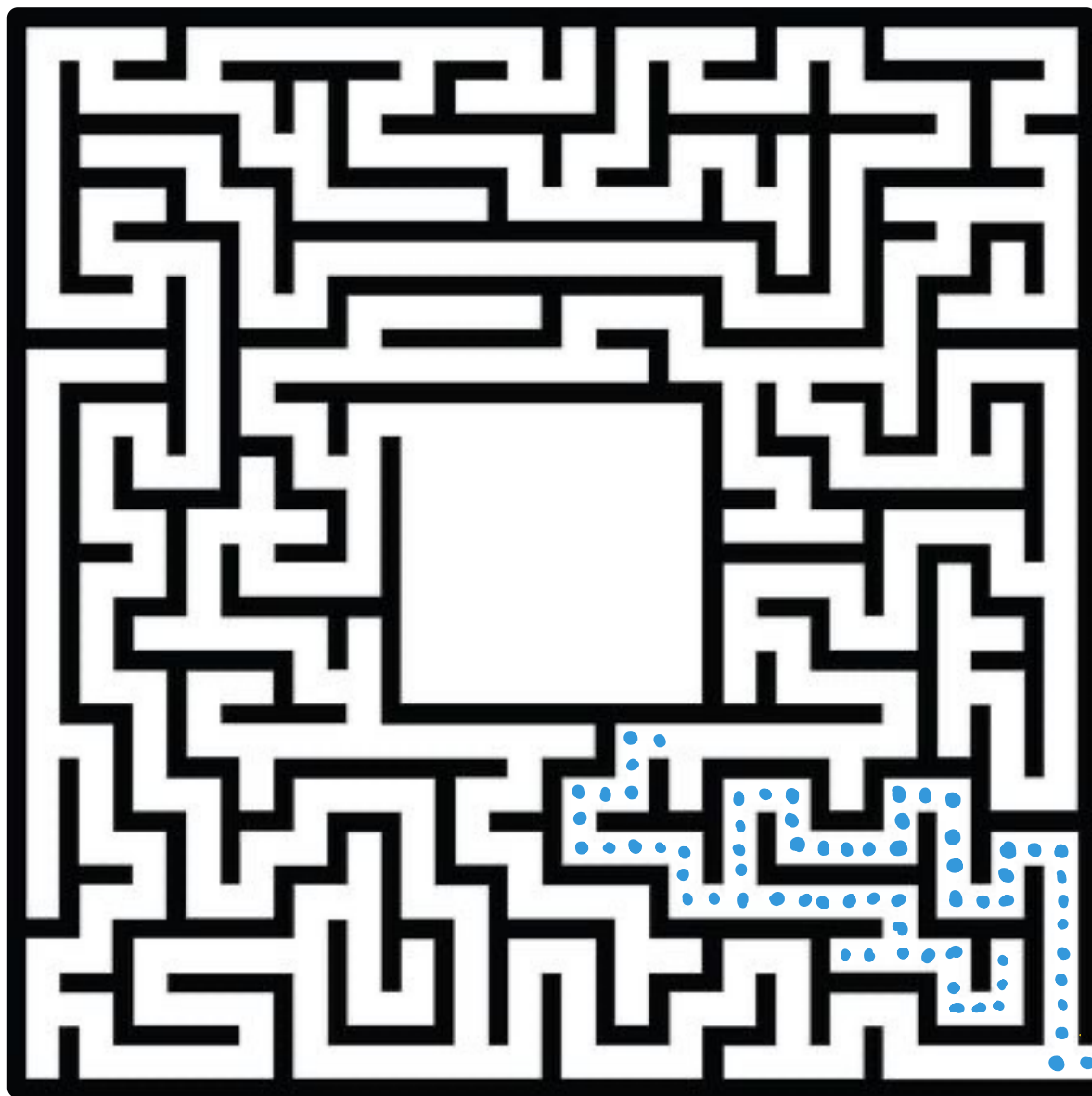
2. for each edge (s, v)

(a) if $\text{recursive-search}(v, t) = \text{TRUE}$

then return TRUE

3. return FALSE





Breadcrumbs

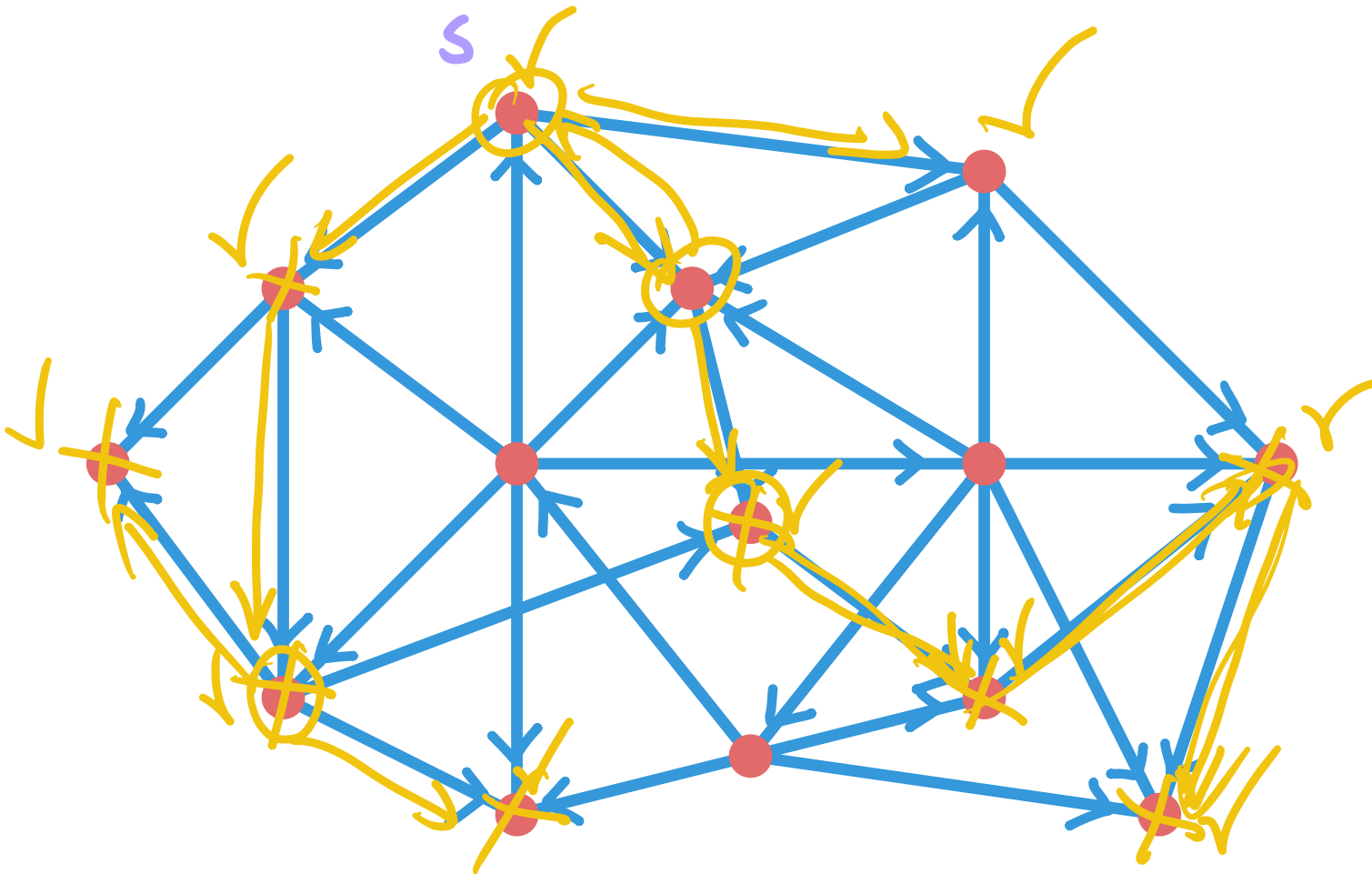
DFS(v)

1. if v is unmarked

A. mark v

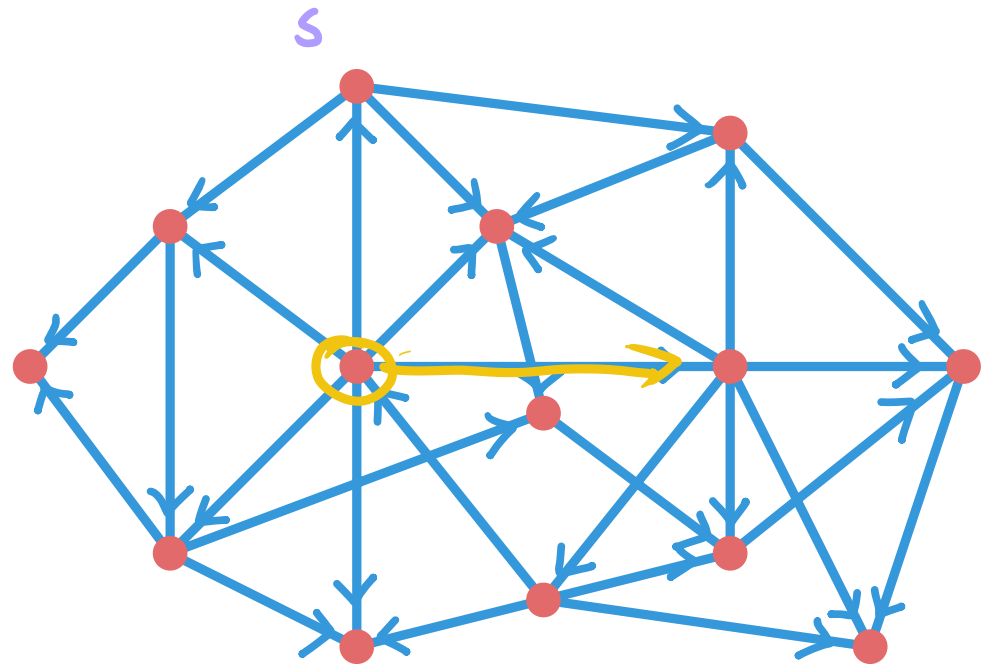
B. for each edge (v, w) from v

1. DFS(w)



DFS(v)

1. if v is unmarked
 - A. mark v
 - B. for each edge (v, w) from v
 1. DFS(w)



2 claims

- ① DFS(v) runs in $O(m+n)$ time
- ② DFS(v) visits all vertices reachable from v

DFS(v)

① DFS(v) runs in $O(m+n)$ time

1. if v is unmarked

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B. for each edge (v, w) from v

1. DFS(w)

Proof.

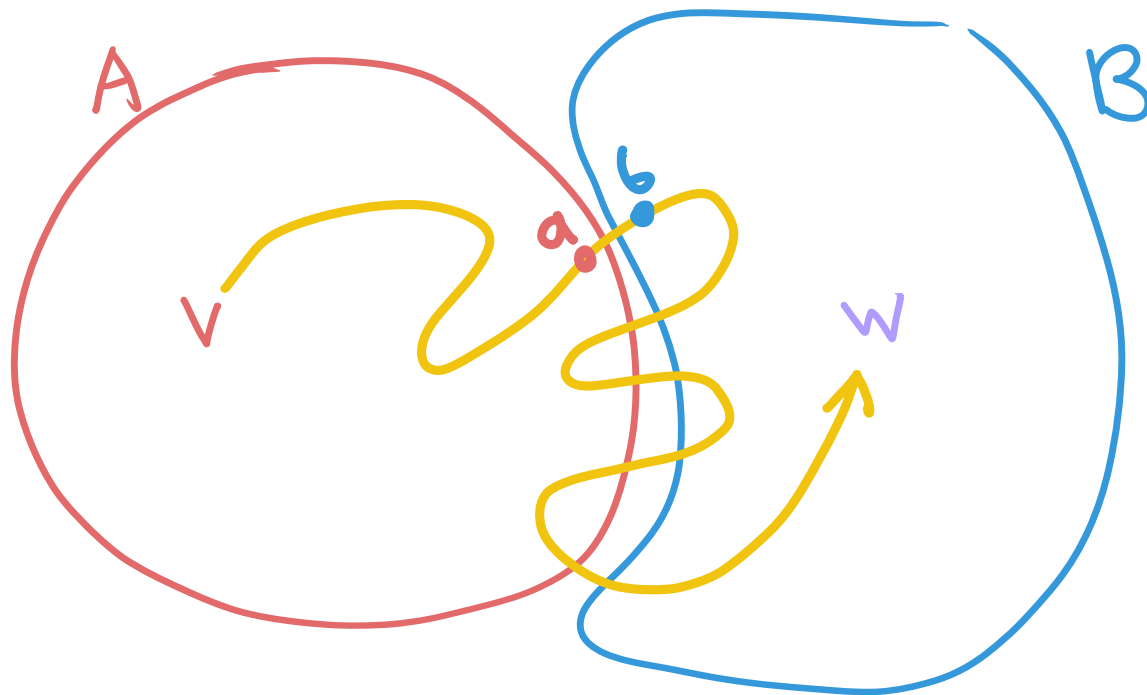
② $\text{DFS}(v)$ visits all vertices reachable from v

$A = \{\text{vertices } a \text{ that are marked}\}$

$B = V \setminus A = \{\text{unmarked vertices}\}$

let w be reachable from v

suppose by contradiction that $w \in B$



$\text{DFS}(v)$

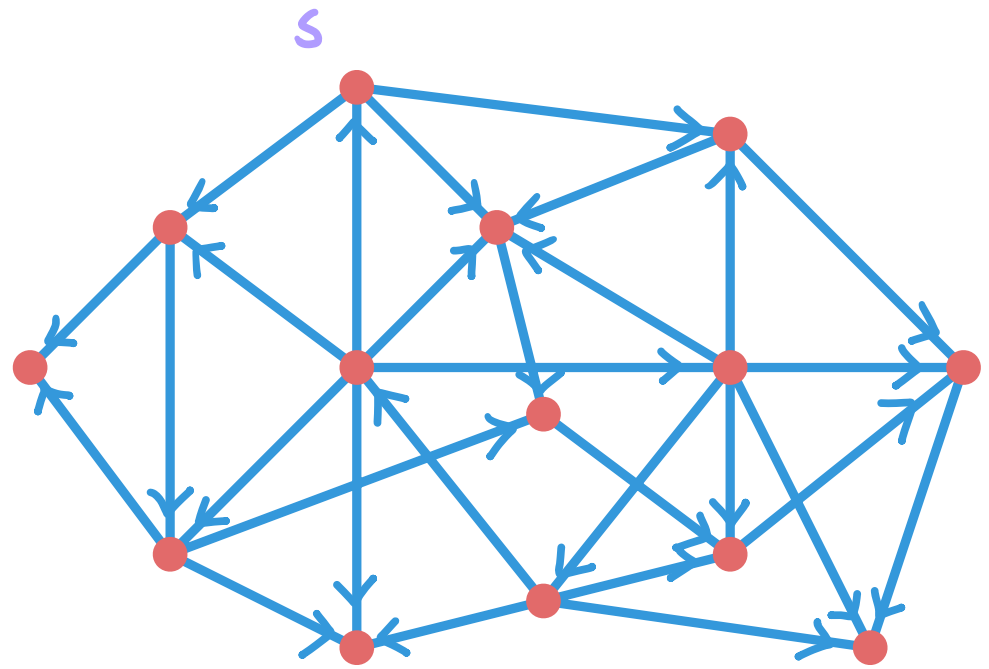
1. if v is unmarked
 - A. mark v
 - B. for each edge (v, w) from v
 1. $\text{DFS}(w)$

consider any path from v to w

since the path starts in A and ends in B , there must be some directed edge from some $a \in A$ and some $b \in B$

DFS(v)

1. if v is unmarked
 - A. mark v
 - B. for each edge (v, w) from v
 1. DFS(w)



2 claims

- ① DFS(v) runs in $O(m+n)$ time
- ② DFS(v) visits all vertices reachable from v

more precisely: DFS(v) returns only after all vertices reachable from v have been marked

2 claims

① DFS(v) runs in $O(m+n)$ time

② DFS(v) visits all vertices reachable from v

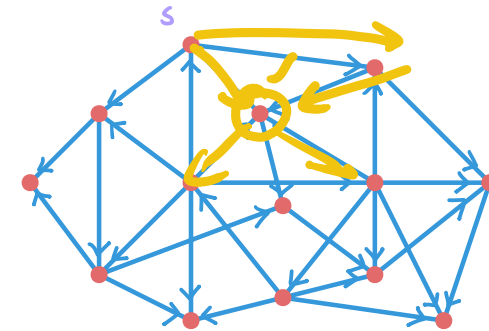
DFS(v)

1. if v is unmarked

A. mark v

B. for each edge (v, w) from v

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I. DAG's and topological orderings

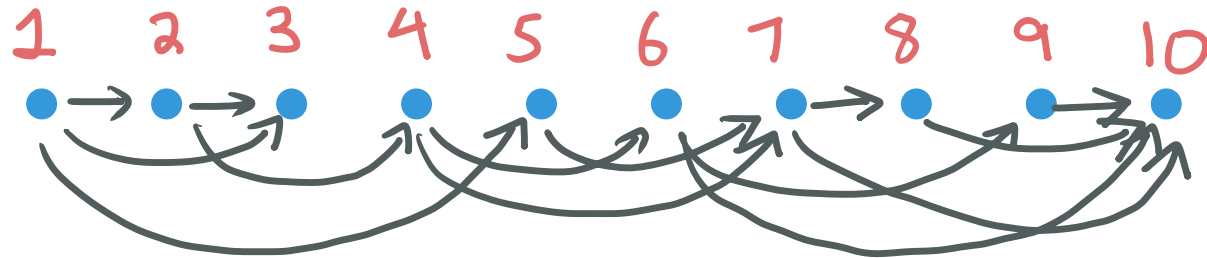
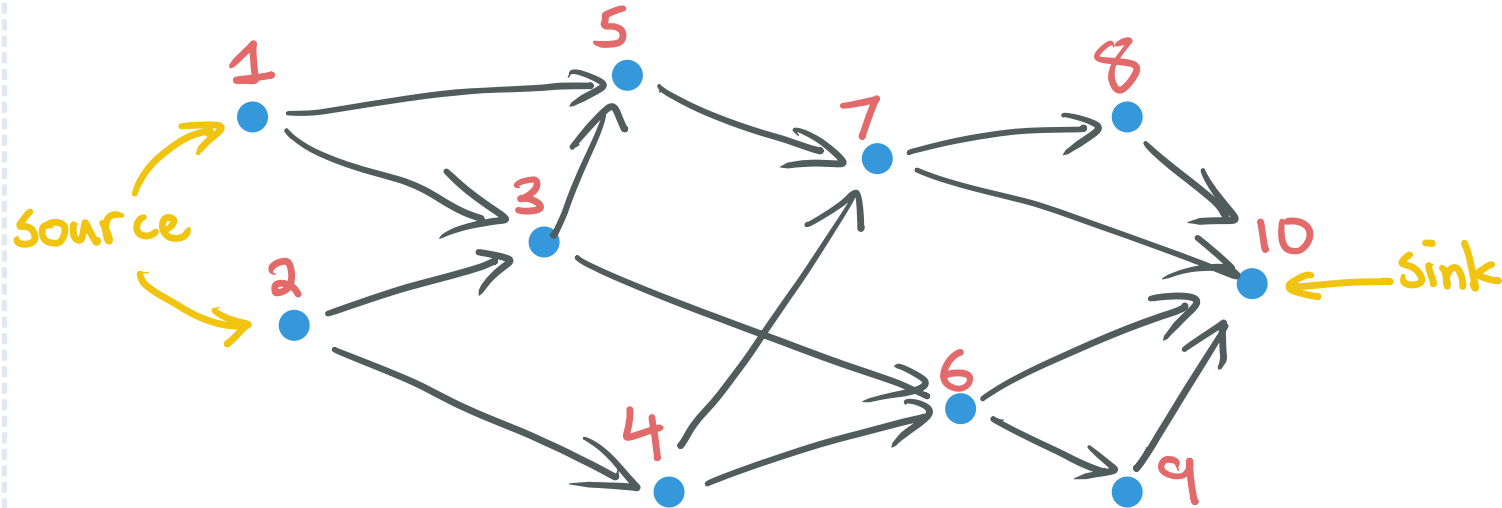
II. Searching graphs

III. Sorting graphs

Recall:

How to topologically sort:
repeatedly remove sources

Topological ordering (of V):
any ordering consistent w/ edges

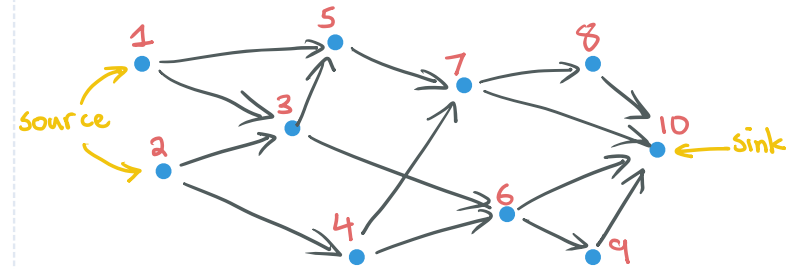


DFS(v)

1. if v is unmarked
 - A. mark v
 - B. for each edge (v, w) from v
 1. DFS(w)

How to topologically sort:
repeatedly remove sources

Topological ordering (of V):
any ordering consistent w/ edges



between the time that

v is marked and DFS(v) exits

DFS only searches vertices reachable from v

Lemma let $v, w \in V$ where

- v can reach w
- w cannot reach v .

DFS(v)

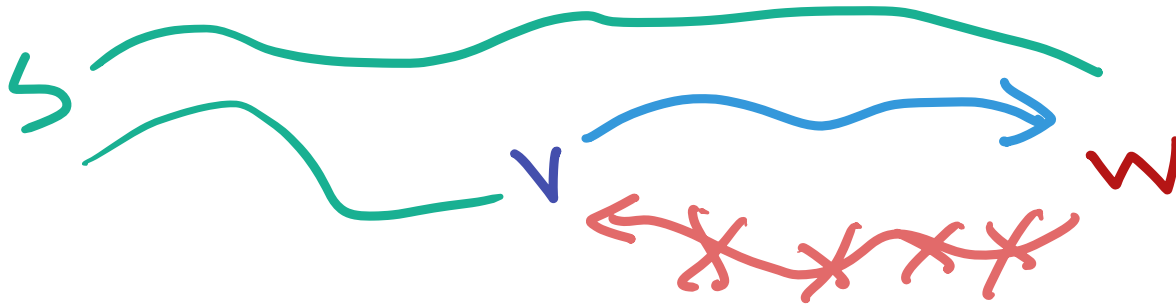
1. if v is unmarked

A. mark v

B. for each edge (v, w) from v

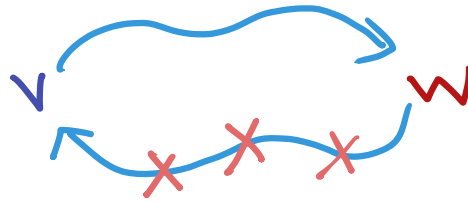
1. DFS(w)

Then any DFS that explores both v & w must
finish DFS(w) before DFS(v)



Lemma let $v, w \in V$ where

- v can reach w
- w cannot reach v .



DFS(v)

1. if v is unmarked
 - A. mark v
 - B. for each edge (v, w) from v
 1. DFS(w)

Then any DFS that explores both v & w must finish DFS(w) before DFS(v)

Proof

Case 1: DFS marks v first

then DFS marks w and

finishes DFS(w) before finishing v .

Case 2: DFS marks w first

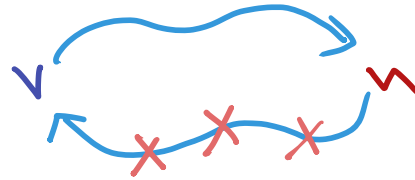
then DFS(w) only searches vertices

reachable from w — which excludes v —

before finishing

Lemma let $v, w \in V$ where

- v can reach w
- w cannot reach v .



Then any DFS that explores both v & w must finish $\text{DFS}(w)$ before $\text{DFS}(v)$

post-DFS order: print out vertices v as we finish $\text{DFS}(v)$

post-DFS(v)

1. If v is unmarked.
 - A. Mark v .
 - B. For each edge (v, w) from v :
 1. post-DFS(w).

/ The following line is the only difference from DFS.*

- C. Append v to the output

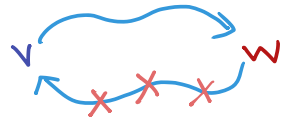
post-DFS-driver($G = (V, E)$)

1. For each vertex $v \in V$
 - A. post-DFS(v)

Lemma let $v, w \in V$ where

- v can reach w

- w cannot reach v .



Then any DFS that explores both v & w must finish $\text{DFS}(w)$ before $\text{DFS}(v)$

post-DFS(v)

1. If v is unmarked.

A. Mark v .

B. For each edge (v, w) from v :

1. $\text{post-DFS}(w)$.

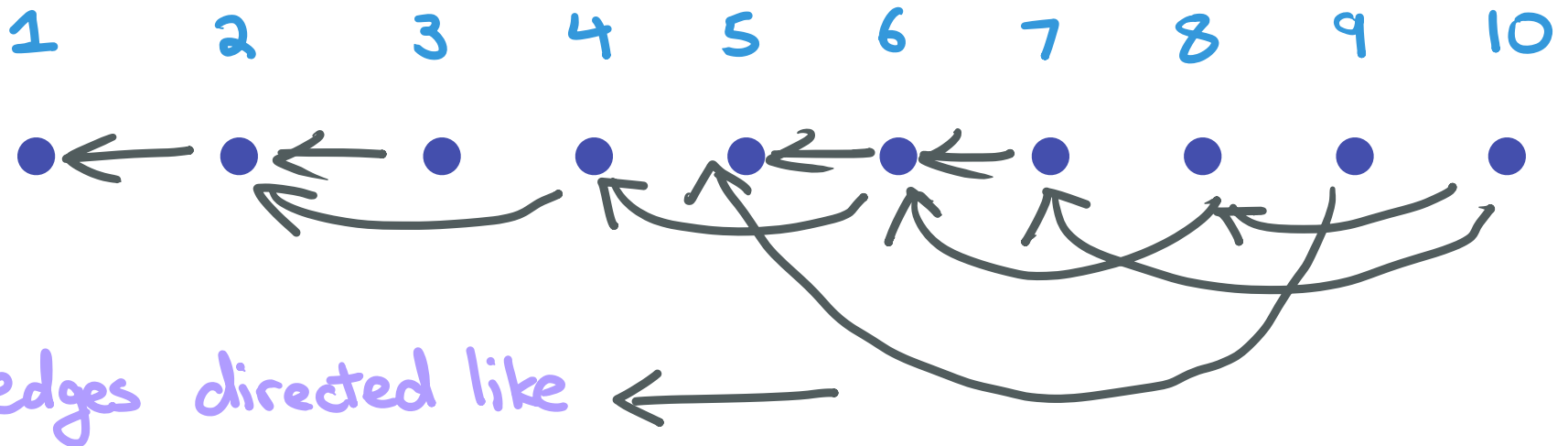
/ The following line is the only difference*

C. Append v to the output

Consider DAG's.

if u can reach v , then v cannot reach u (why?)

so DFS prints v before u



all edges directed like

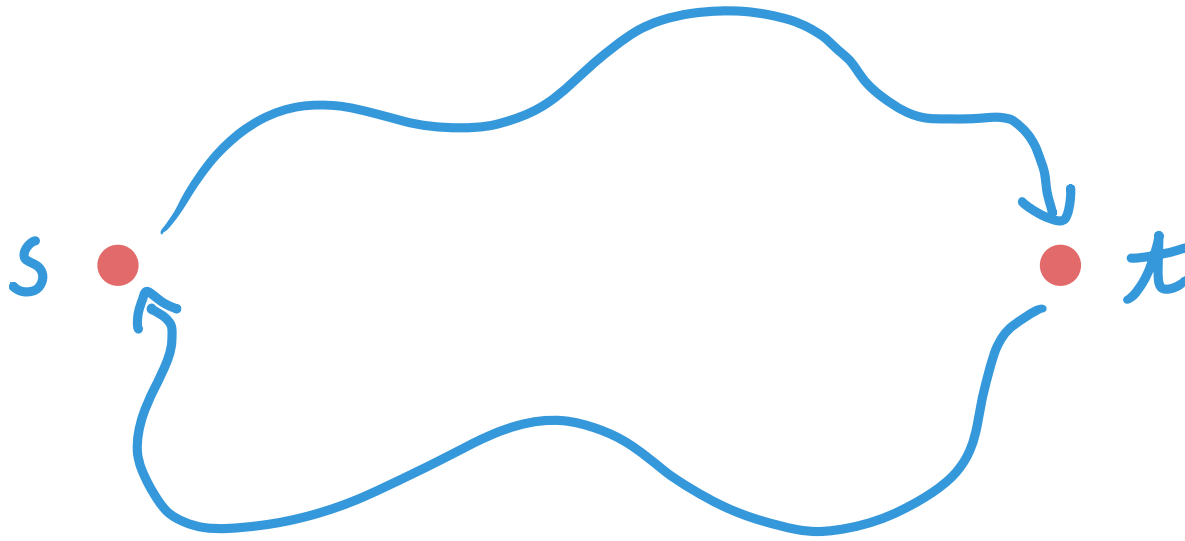
Recall:

Theorem 6.1. *Given a DAG G with m edges and n vertices, one can compute a topological ordering of V in $O(m + n)$ time.*

DFS (+ reversing) gives a second algo.

Strong connectivity

two vertices s and t are strongly connected
if s can reach t and t can reach s

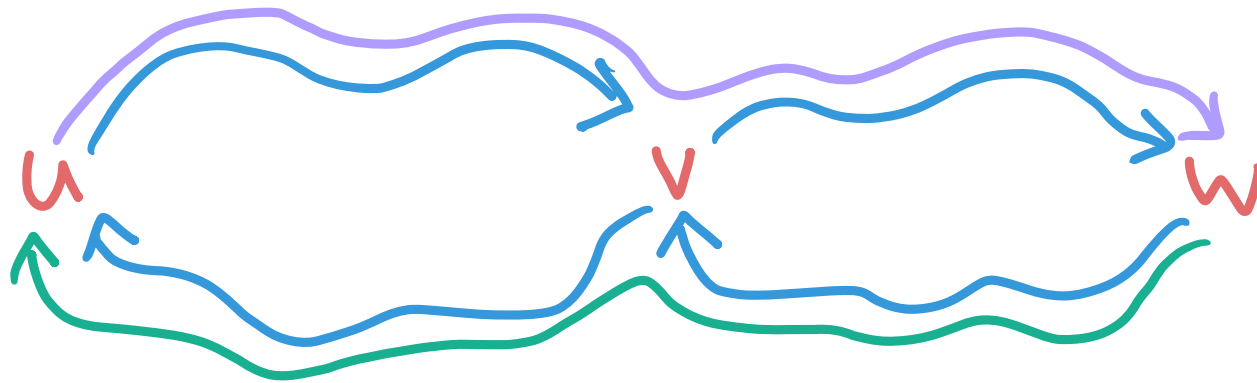


a graph is strongly connected if every pair
of vertices is strongly connected

Claim strong connectivity defines an equivalence class

(that is, if u is strongly connected to v , and v is strongly connected to w , then u is strongly connected to w)

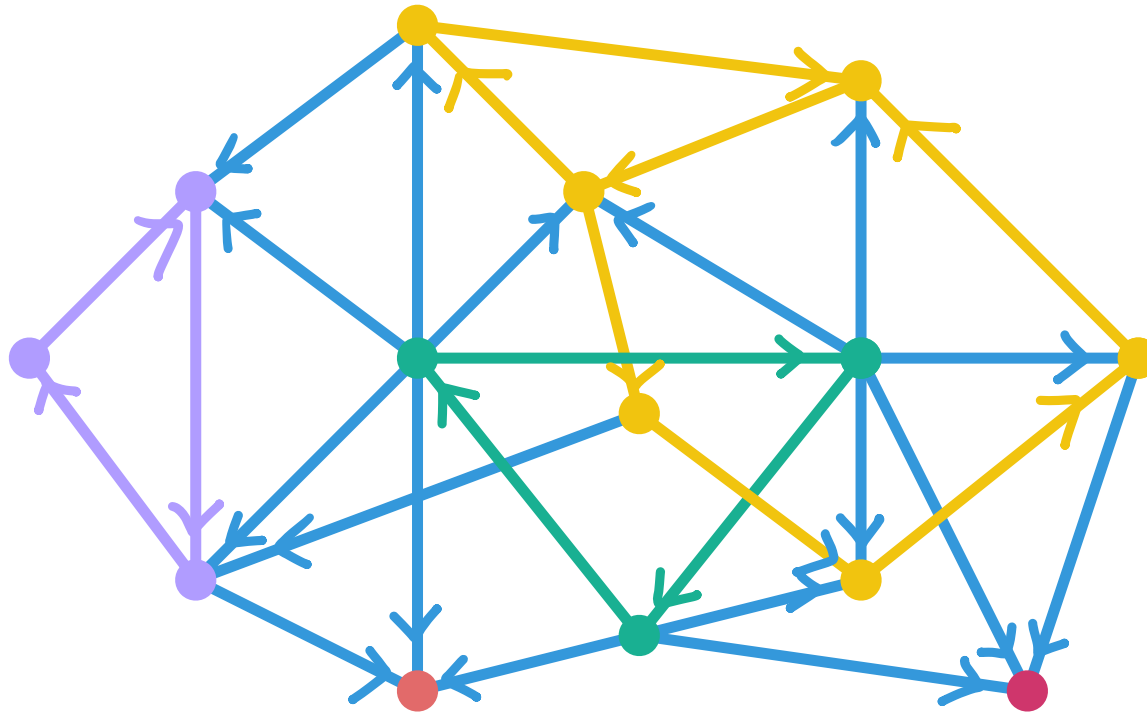
Proof



combine paths from $u \rightsquigarrow v$, $v \rightsquigarrow w$ to get path
from $u \rightsquigarrow w$

similar for other direction

each equivalence class of strongly connected vertices is called a strongly connected component (SCC)



Goal: find the SCC's algorithmically
(strongly connected components)

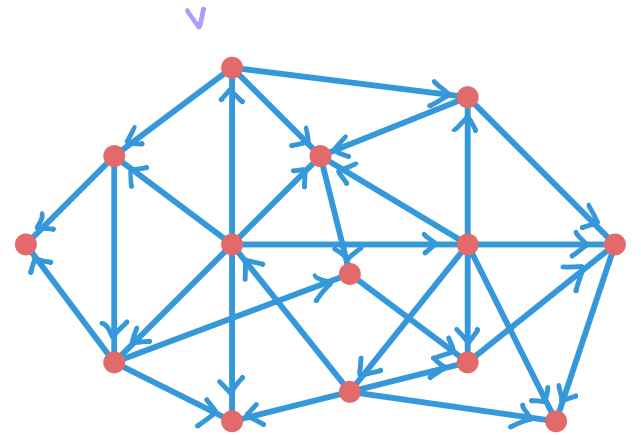
First:

given v , find the SCC of v

(strongly
connected
component)

ideas?

1. DFS(v)



First:

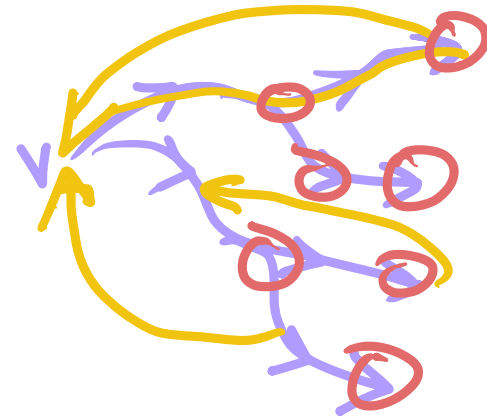
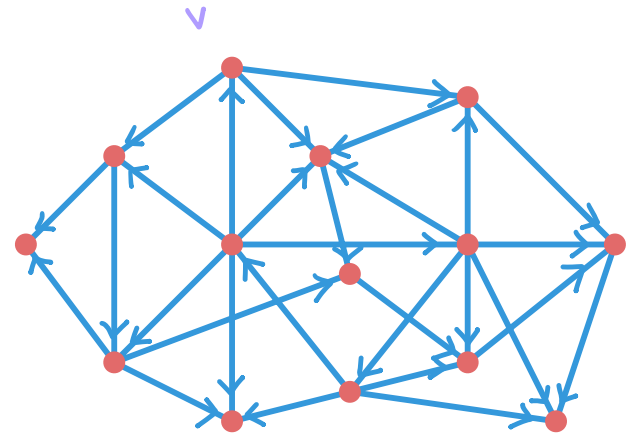
given v , find the SCC of v
(strongly connected component)

① run DFS from v

② * run DFS from v in the reversed graph

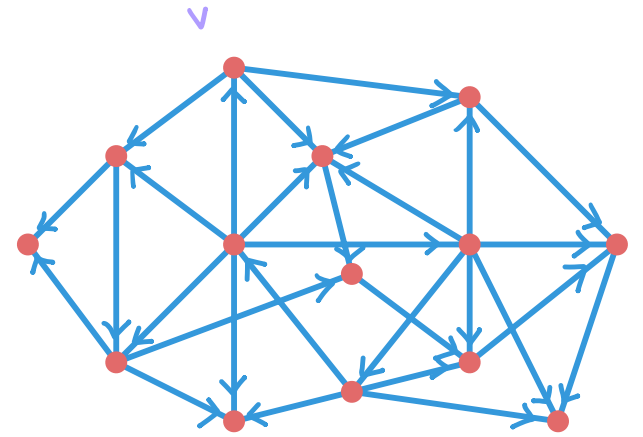
③ return set of vertices visited in both DFS

* ② visits all vertices that can reach v in original graph



First:

given v , find the SCC of v
(strongly connected component)



① run DFS from v

②* run DFS from v in the reversed graph

③ return set of vertices visited in both DFS



*② visits all vertices that can reach v in original graph

$SCC(v)$

① run DFS from v

$O(m+n)$

② run DFS from v in
the reversed graph

$O(m+n)$

③ return set of vertices
visited in both DFS

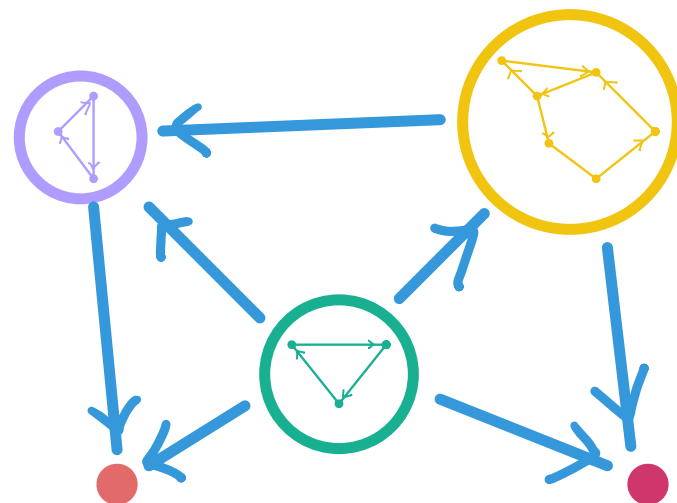
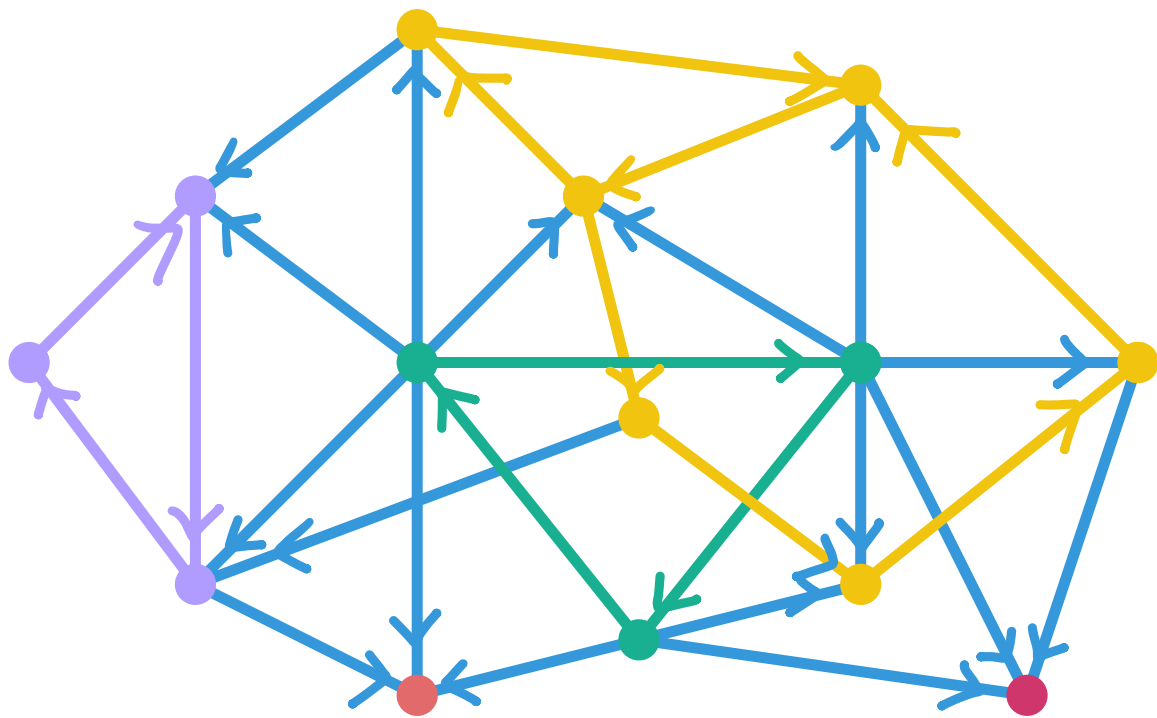
$O(m+n)$

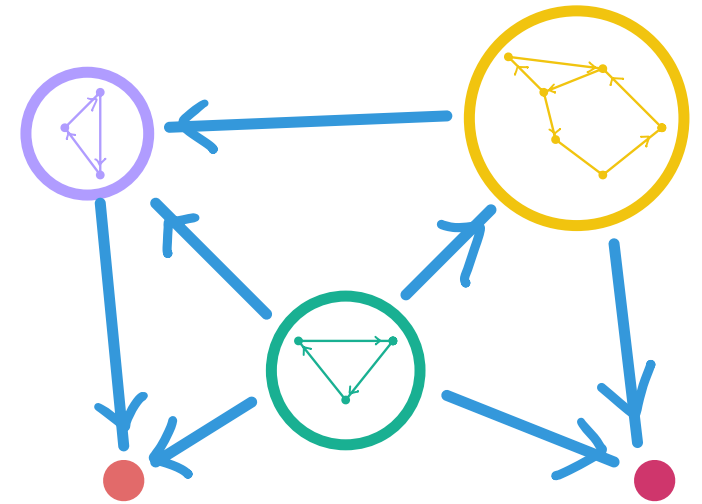
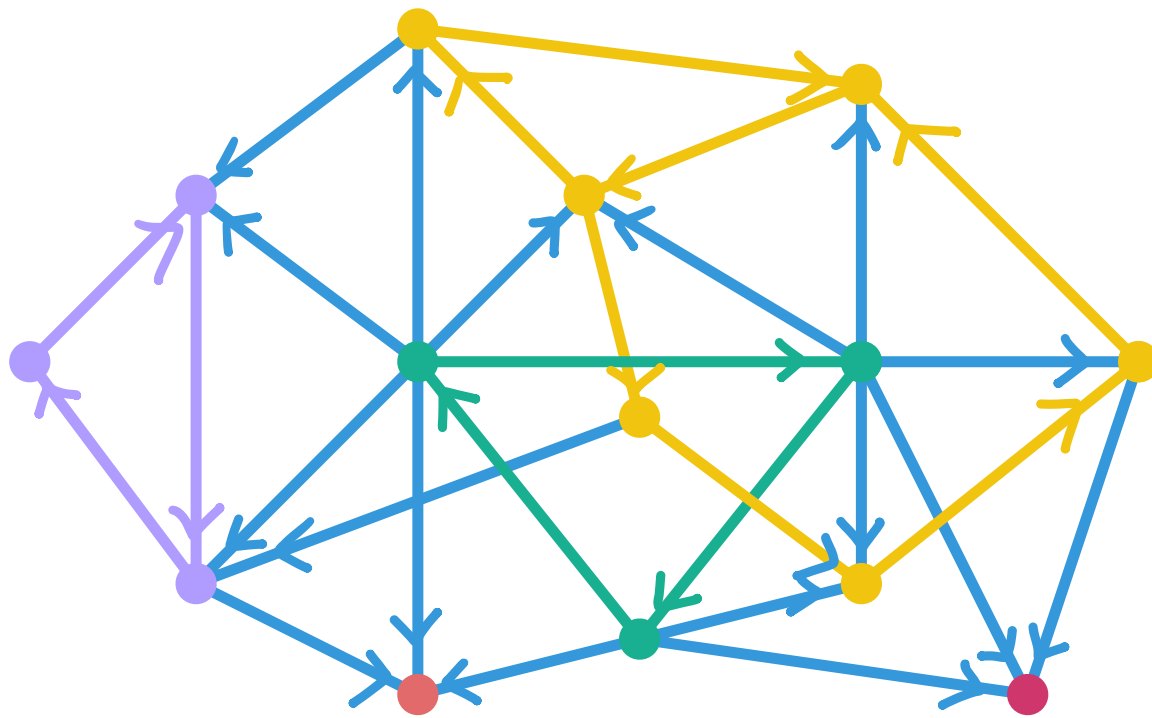
to get all the SCC's:

run from every vertex

$$n \times O(m+n) = O(mn + n^2)$$

better?





Conspicuously: no cycles a DAG!
 (directed acyclic graph)

Our current approach

Slow-SCC's ($G=(V,E)$)

1. until $V=\emptyset$

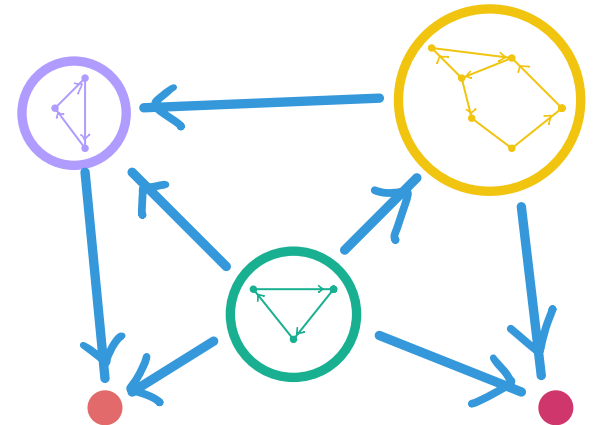
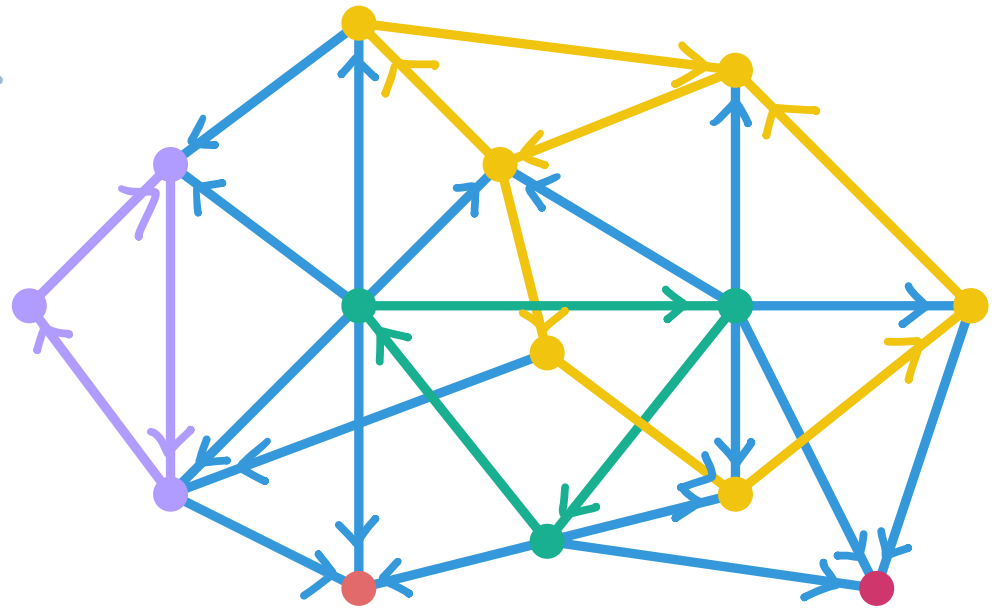
a. let v be any vertex in V

b. $C \leftarrow \text{SCC}(v)$

c. label each vertex in C w/
(some label for C)

d. remove C and all incident
edges from G

(this takes $O(mn)$)



Slow-SCC's ($G=(V,E)$)

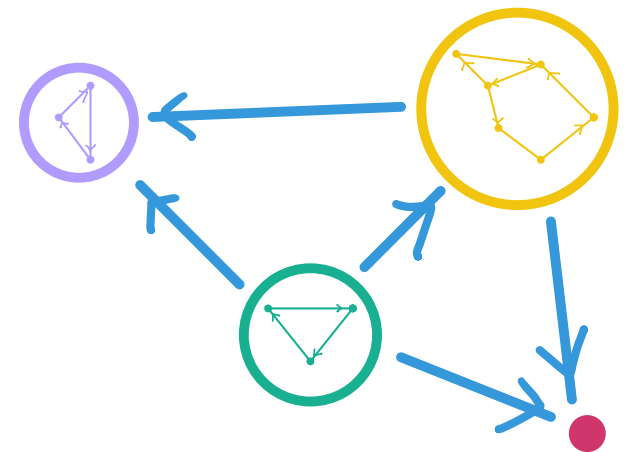
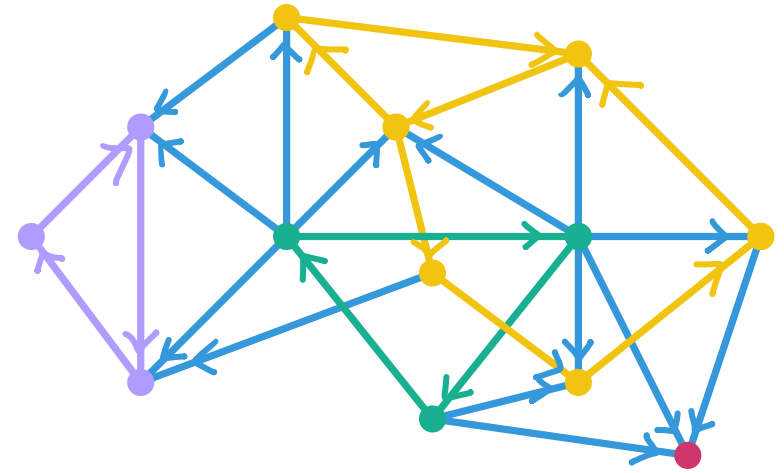
until $V=\emptyset$

let v be any vertex ~~in V~~
in a sink component

$C \leftarrow \text{SCC}(v)$

label each vertex in C w/
(some label for) C

remove C and all incident
edges from G



sink-first

~~slow~~ - SCC's ($G=(V,E)$)

we don't know
how to do this
yet

until $V=\emptyset$

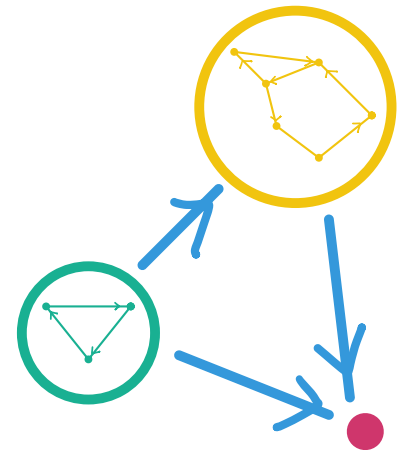
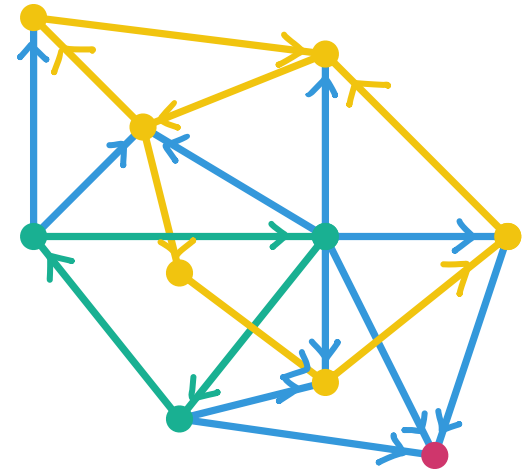
let v be any vertex ~~in V~~
in a sink component

DFS

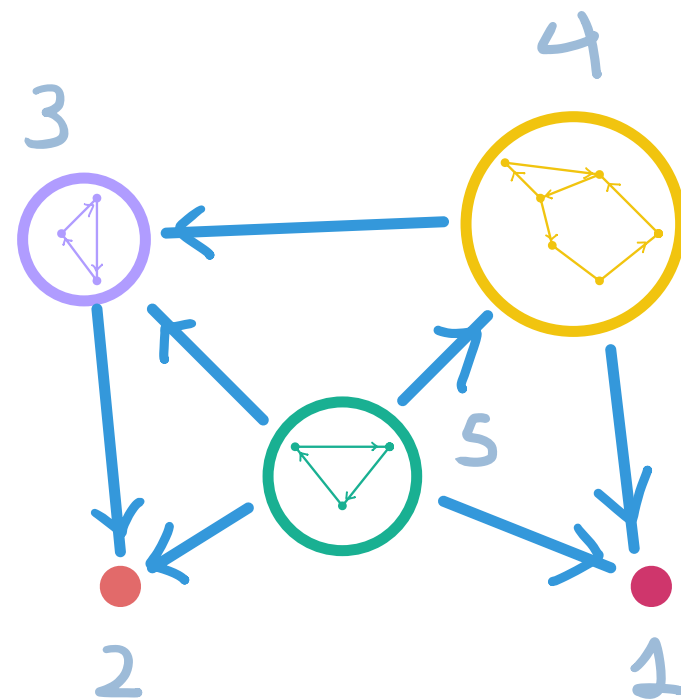
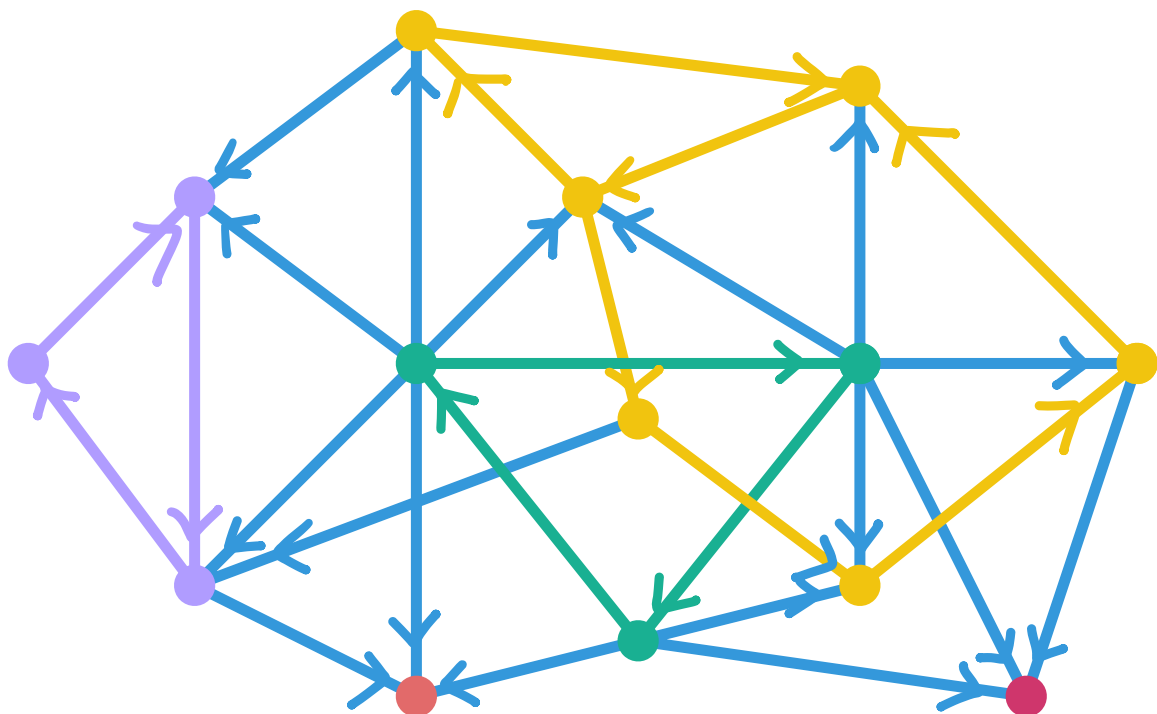
$C \leftarrow \text{SCC}(v)$ // DFS cannot leave sink
label each vertex in C w/ component

(some label for) C

remove C and all incident
edges from G

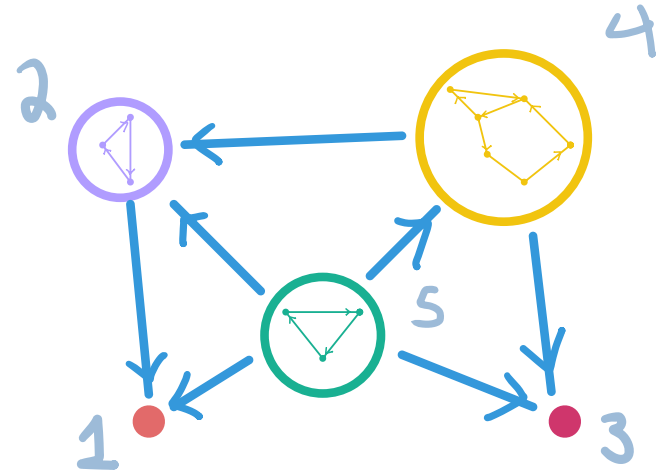
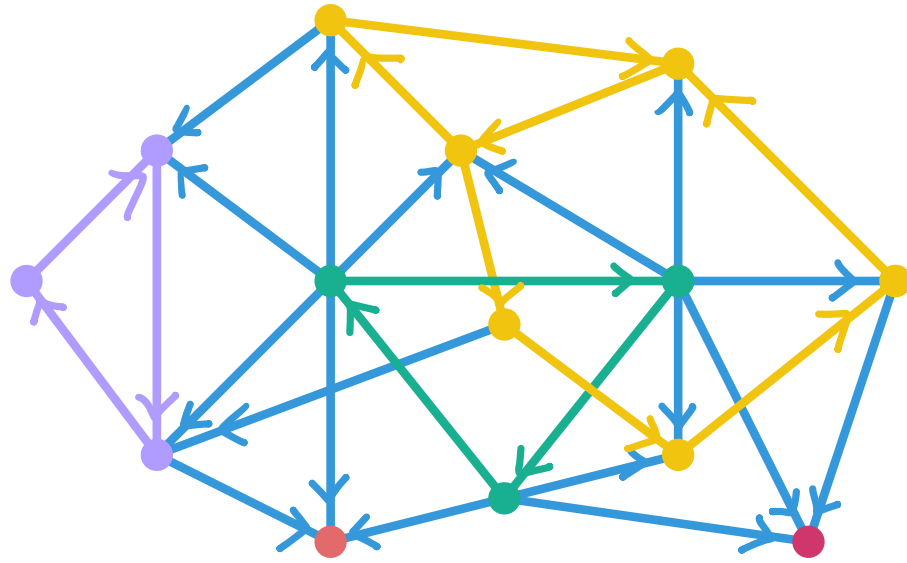


we only visit each vertex once!



To wit:

if we can figure out "sink components" on the fly, then



(a) only 1 DFS per v (instead of reversing)

(b) $O(m+n)$ time total:

each DFS(v) does not leave v 's ^(strongly connected component) SCC

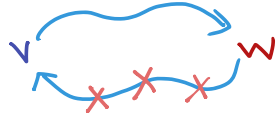
how to find "sink-component" ordering?

post-order DFS order

Lemma let $v, w \in V$ where

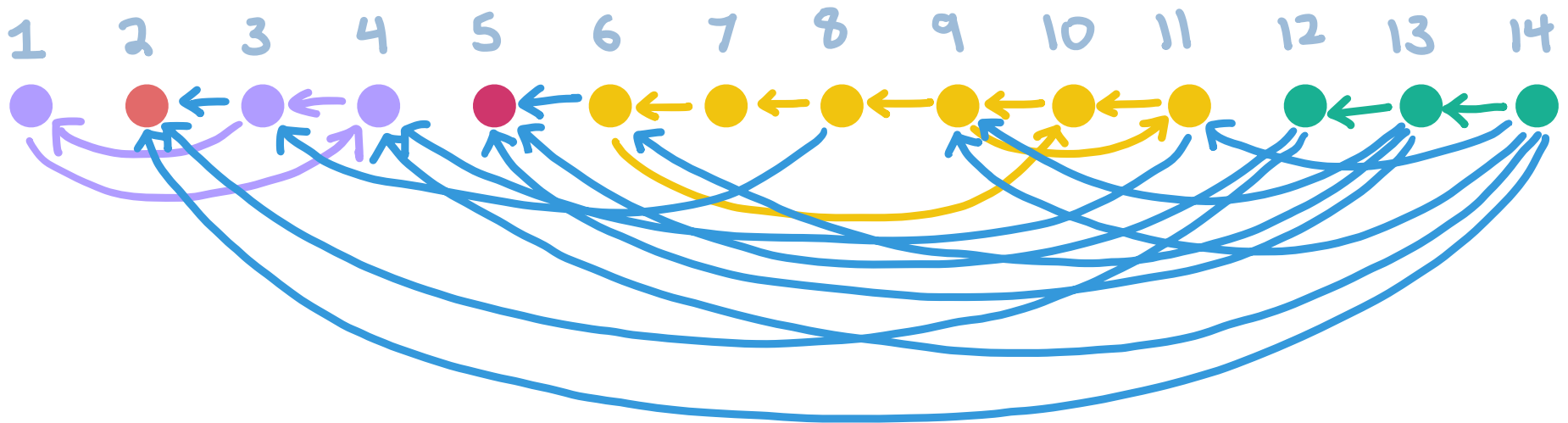
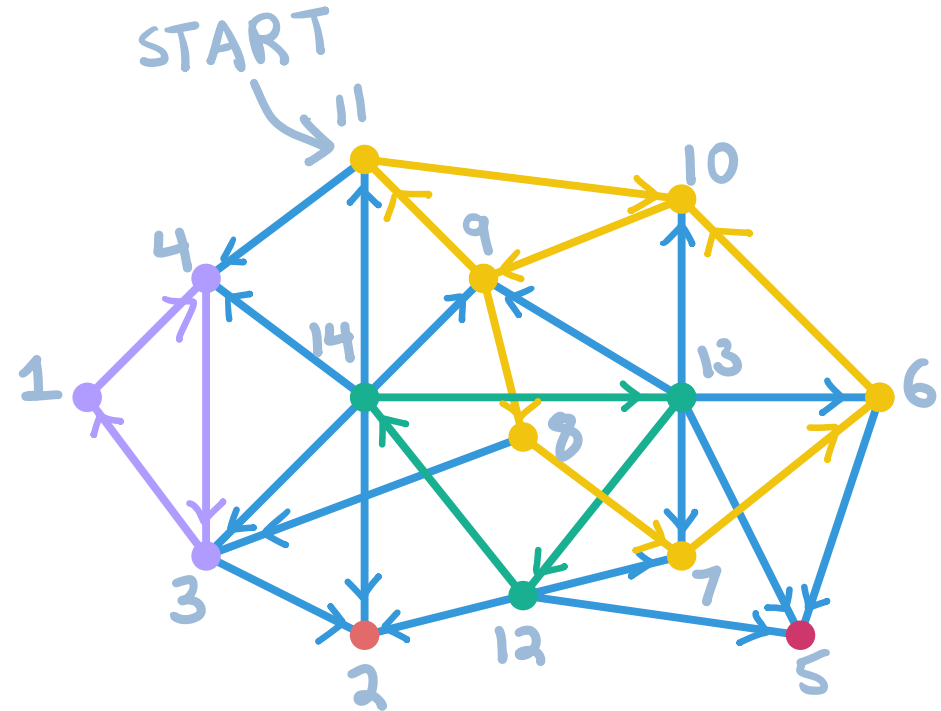
- v can reach w

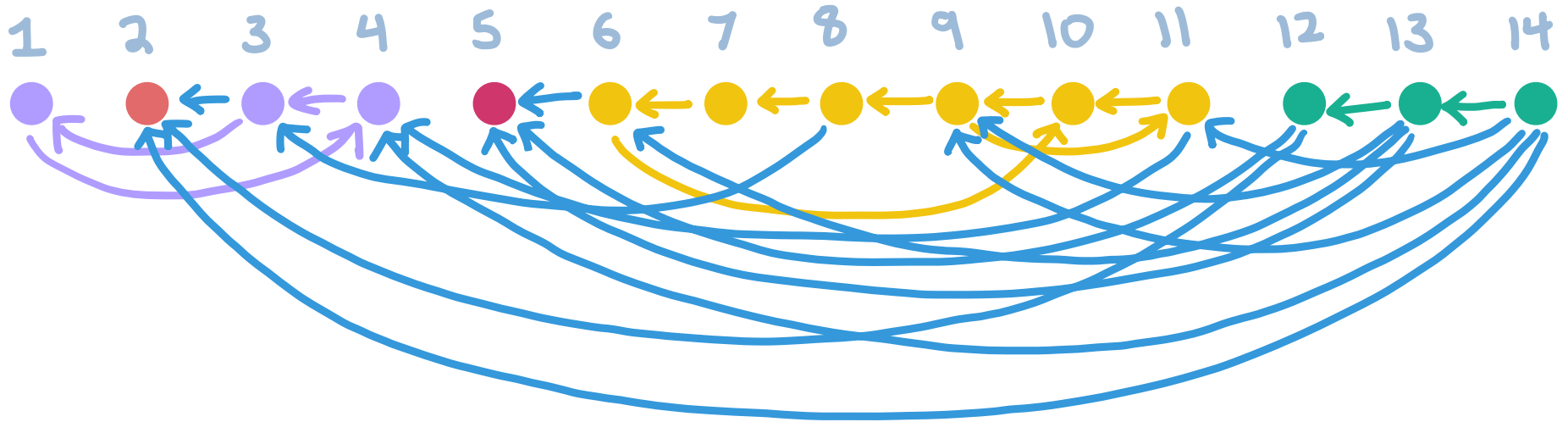
- w cannot reach v .



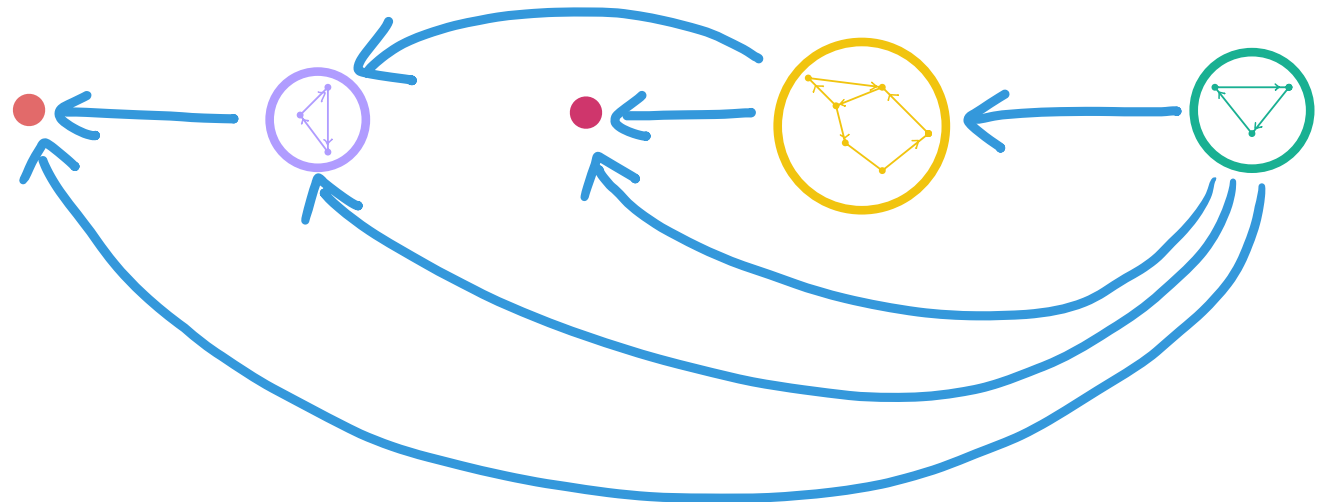
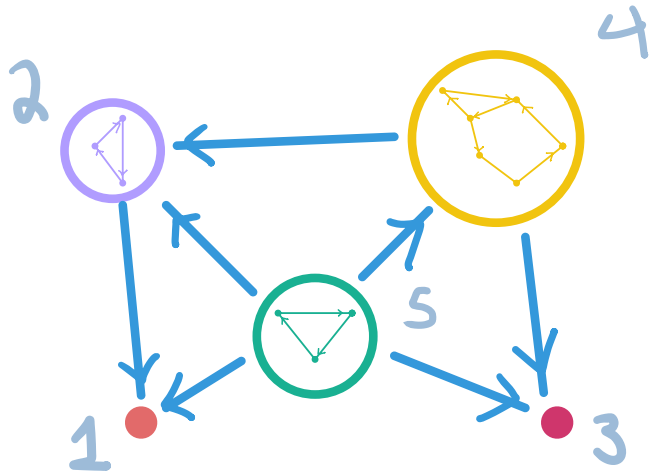
Then any DFS that explores both v & w
must finish $\text{DFS}(w)$ before $\text{DFS}(v)$

↑ i.e. $w \prec v$ in post-DFS order

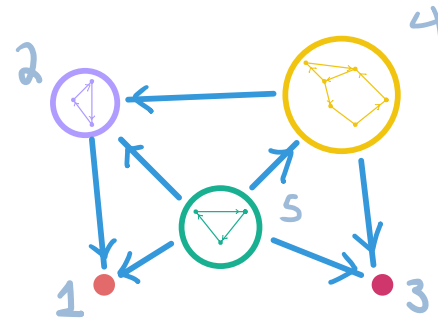
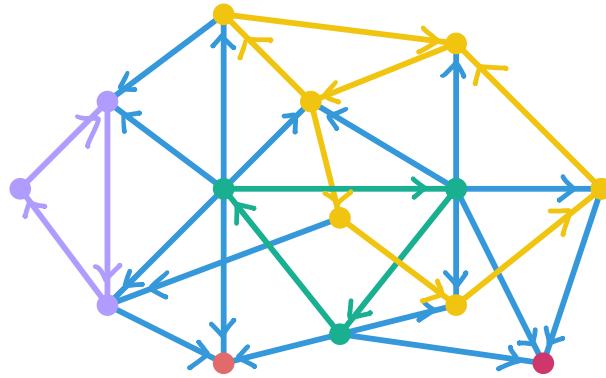




all cross-component edges [in BLUE] are in the same direction



if we can figure out "sink components" on the fly, then



post-order DFS order

Lemma let $v, w \in V$ where

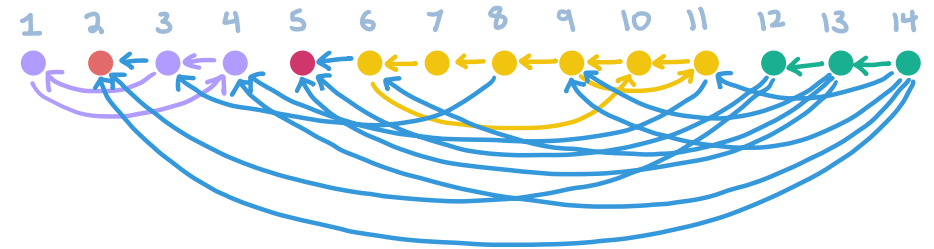
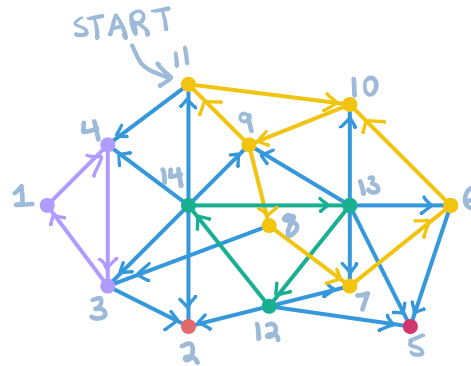
- v can reach w

- w cannot reach v .

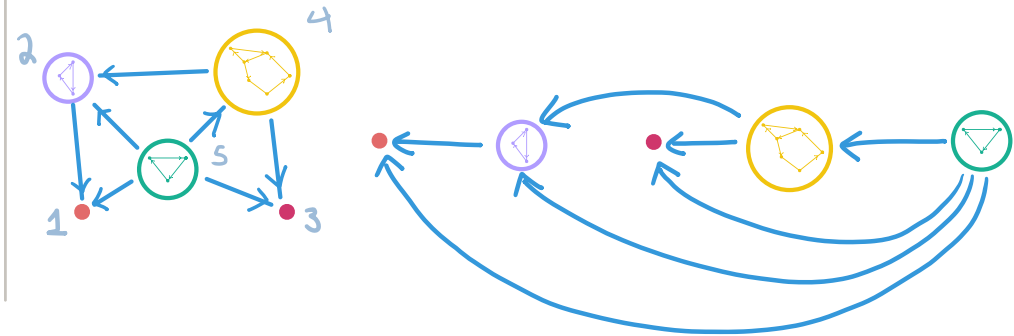
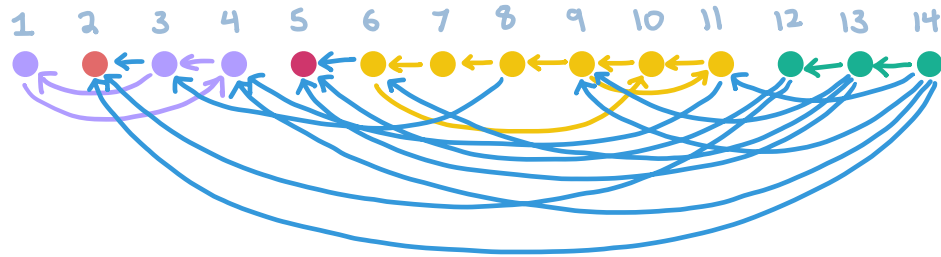


Then any DFS that explores both v & w must finish $\text{DFS}(w)$ before $\text{DFS}(v)$

↑ i.e. $w < v$ in post-DFS order



all cross-component edges [in BLUE] are in the same direction

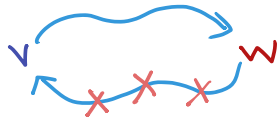


how to find "sink-component" ordering?

Lemma let $v, w \in V$ where

- v can reach w

- w cannot reach v .



Then any DFS that explores both v & w must finish $\text{DFS}(w)$ before $\text{DFS}(v)$

post-DFS(v)

1. If v is unmarked.

A. Mark v .

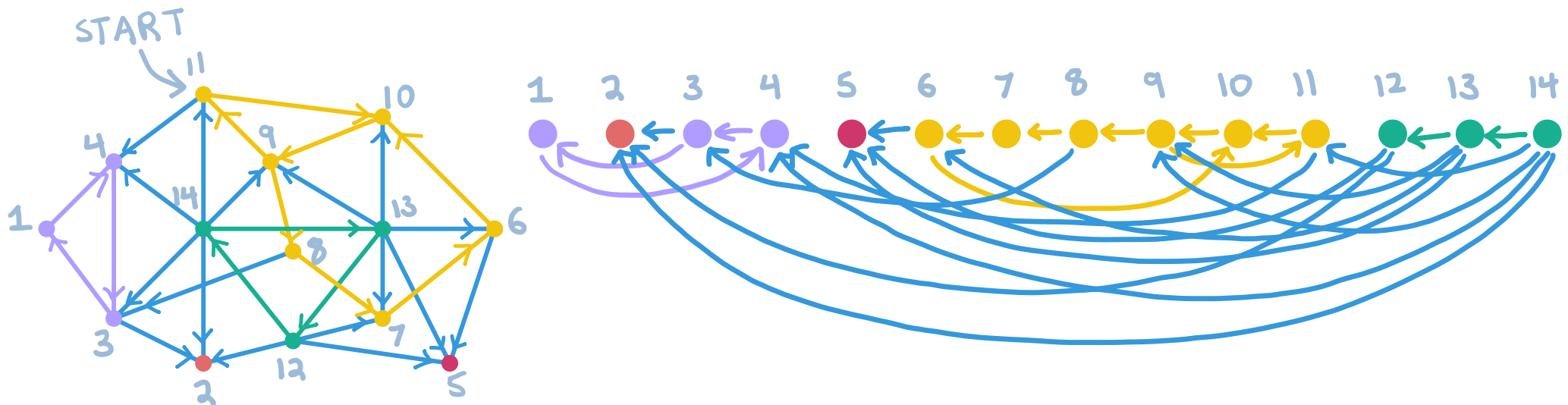
B. For each edge (v, w) from v :

1. $\text{post-DFS}(w)$.

/ The following line is the only difference*

C. Append v to the output

Lemma. let $v_1, \dots, v_n$ list vertices s.t. if v_i can reach v_j , and v_j cannot reach v_i , then $i > j$.
then v_n is from a source component



sink-first

~~slow~~ - SCC's ($G=(V,E)$)

until $V=\emptyset$

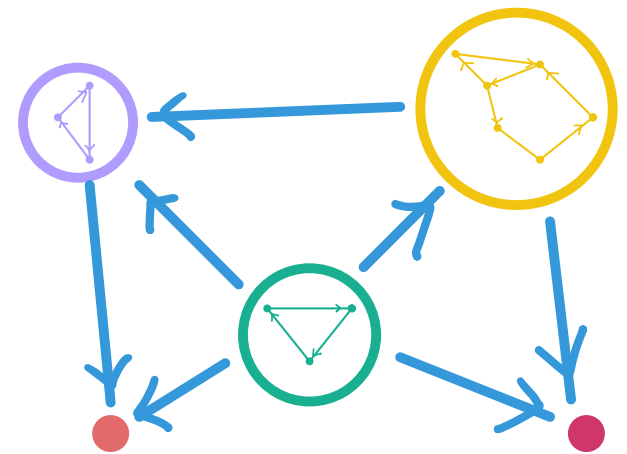
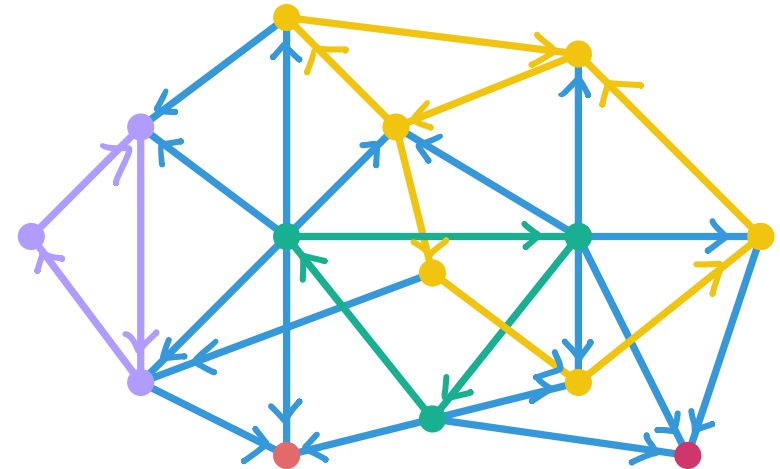
let v be a vertex in a sink component

$C \leftarrow \text{DFS}(v)$ // DFS cannot leave sink component

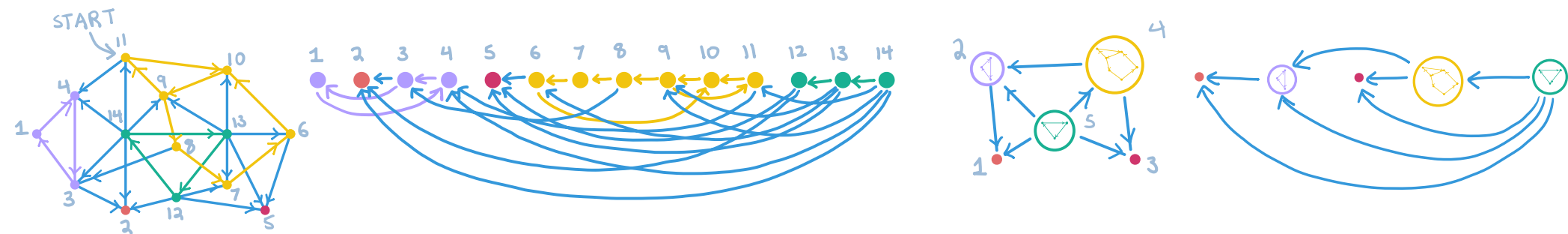
label each vertex in C w/
(some label for) C

remove C and all incident
edges from G

we don't know
how to do this
yet



Lemma. let $v_1, \dots, v_n$ list vertices s.t. if v_i can reach v_j ,
and v_j cannot reach v_i , then $i > j$.
then v_n is from a source component



sink-first-SCC($G = (V, E)$)

1. Compute a post-DFS ordering of V in the *reversed* graph G_r
2. Until $V = \emptyset$
 - A. Let v be the *last* (remaining) vertex in V in the post-DFS order of G_r
// v is in a sink component of G .
 - B. $C \leftarrow$ all vertices marked by $\text{dfs}(v)$
 - C. Append C to the output
 - D. Remove C and all incident edges from G

$O(m+n)$ [!!!]