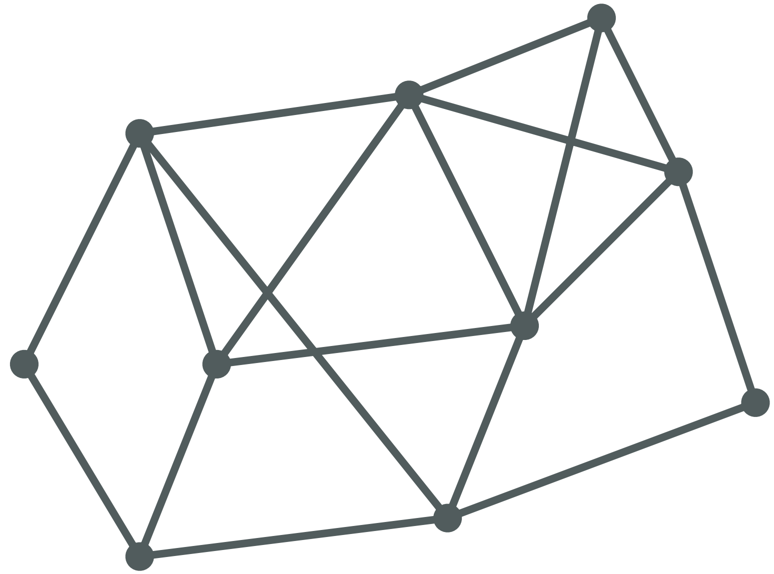
Graph Coloring & Independent Set

Graph Coloring



Goal: color vertices so that no adjacent vertices have the same color, w/ minimum # of colors.

decision version: does k colors suffice?

In "k-coloring problem", k is a fixed constant

Theorem: 3-coloring is NP-hard.

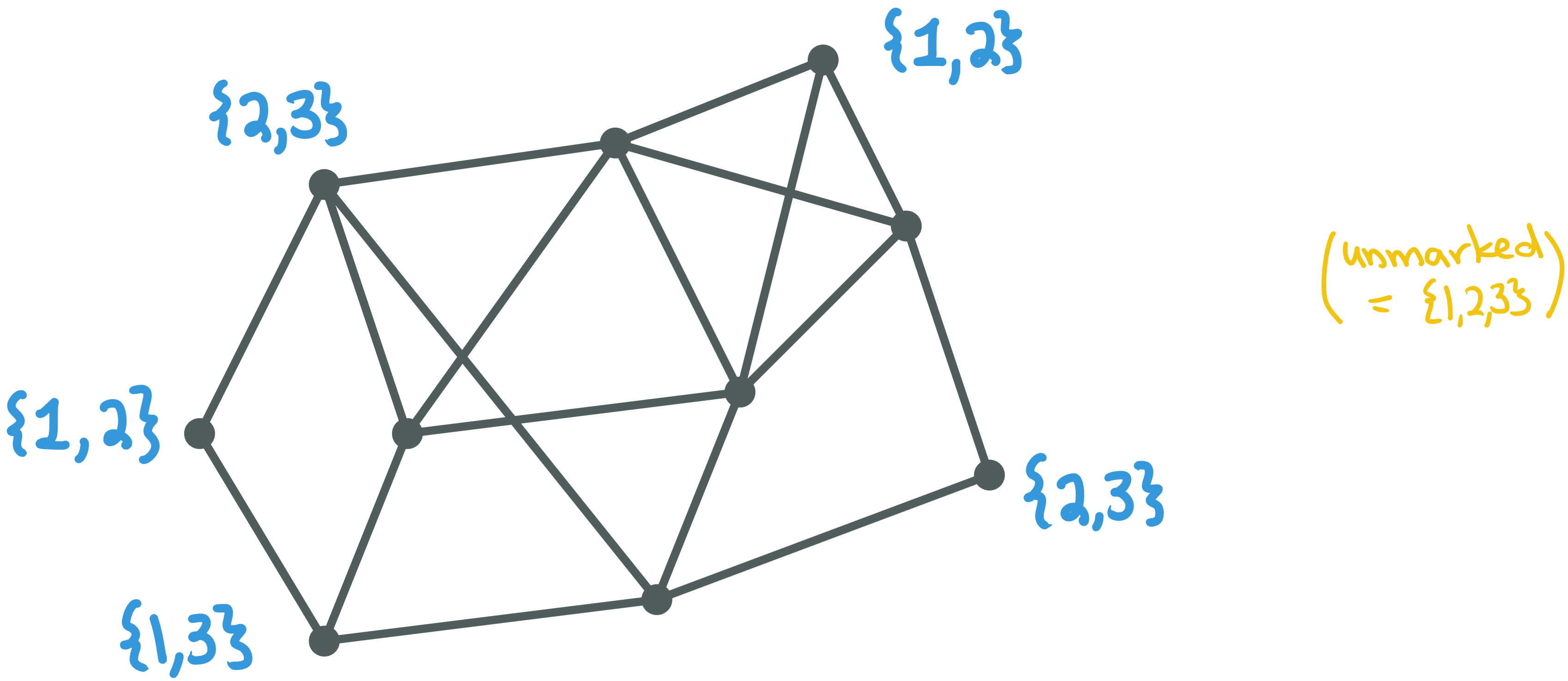
Theorem: 3-coloring is NP-hard.

High-level approach:

3SAT → "subset 3-coloring" → 3-coloring

Subset 3-coloring

for each vertex v, subset $S_v \subseteq \{1,2,3\}$ of colors allowed for v.



3SAT → "subset 3-coloring" → 3-coloring

claim: polytime algo for 3-coloring implies polytime algo for subset 3-coloring.

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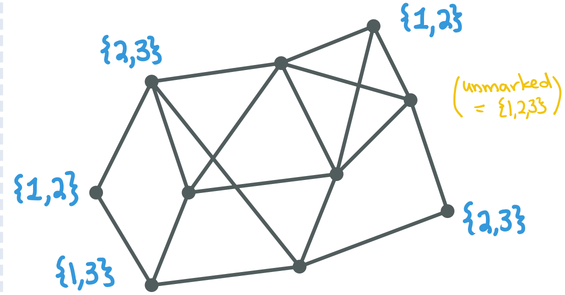

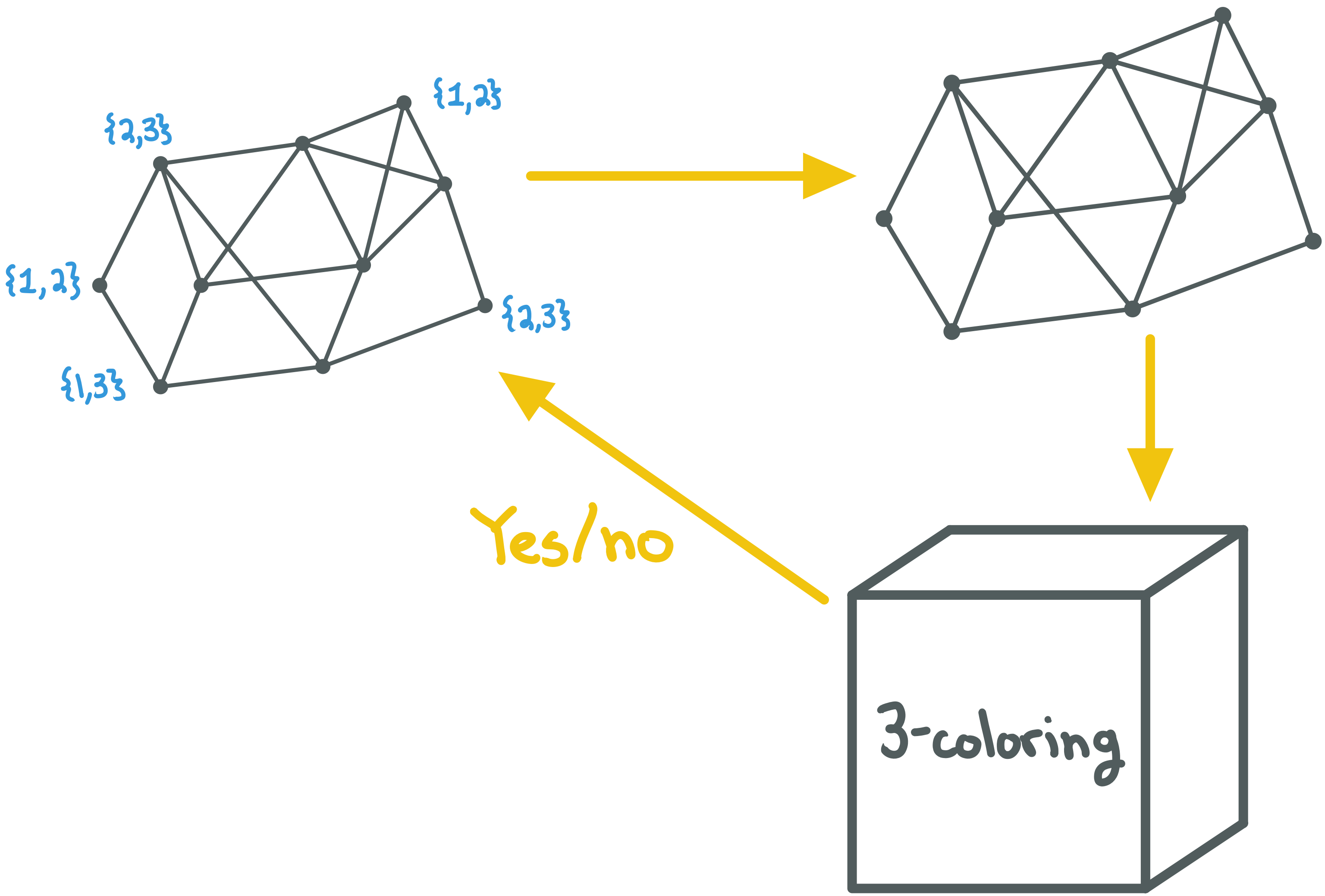

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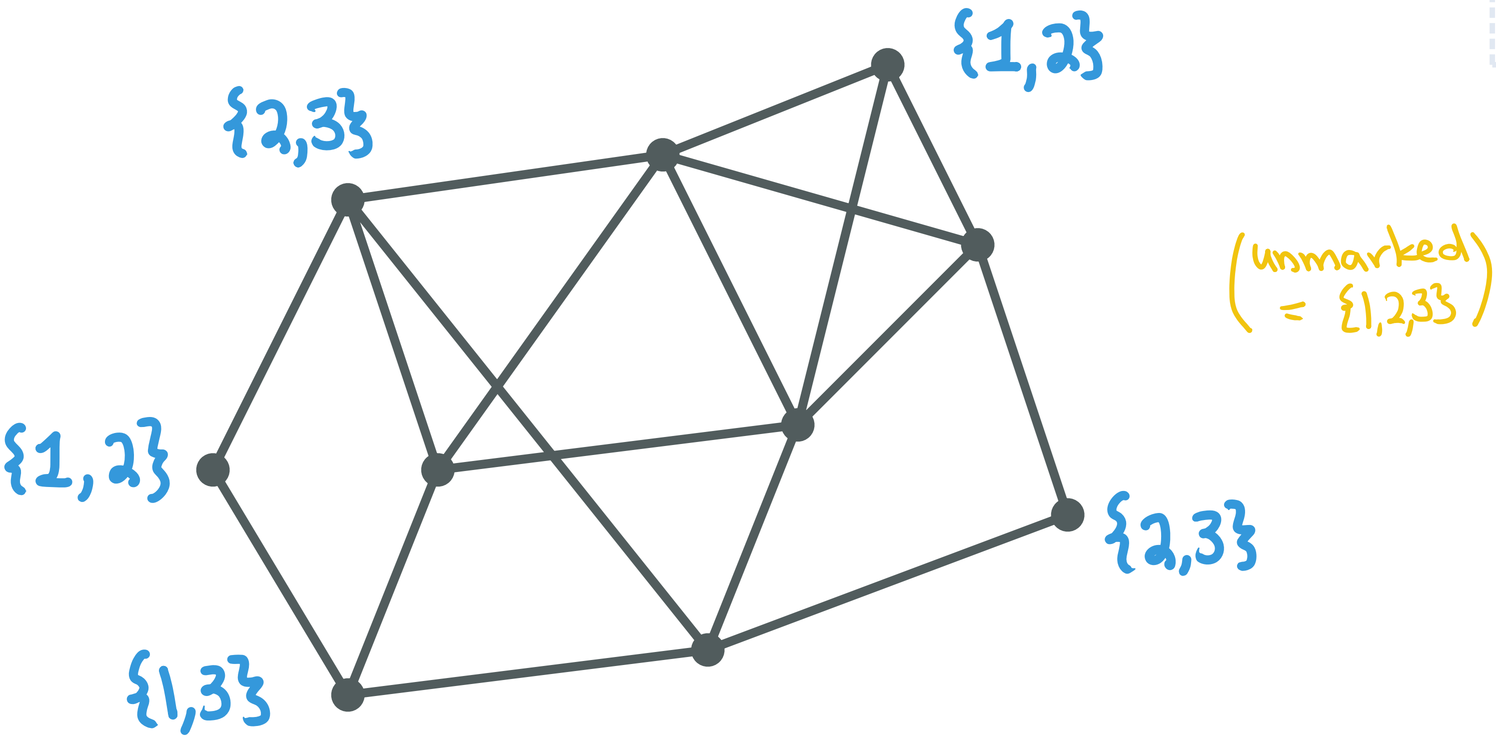
ideas?



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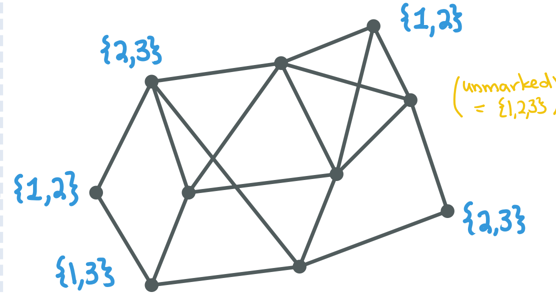

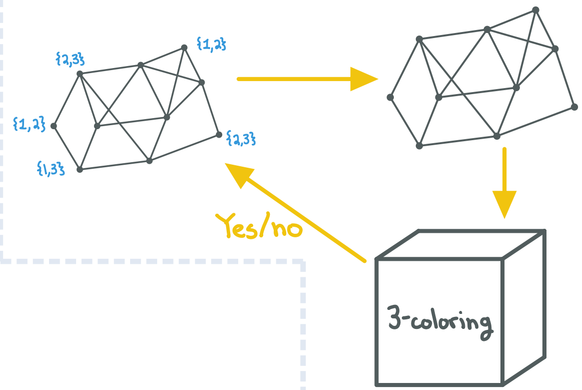

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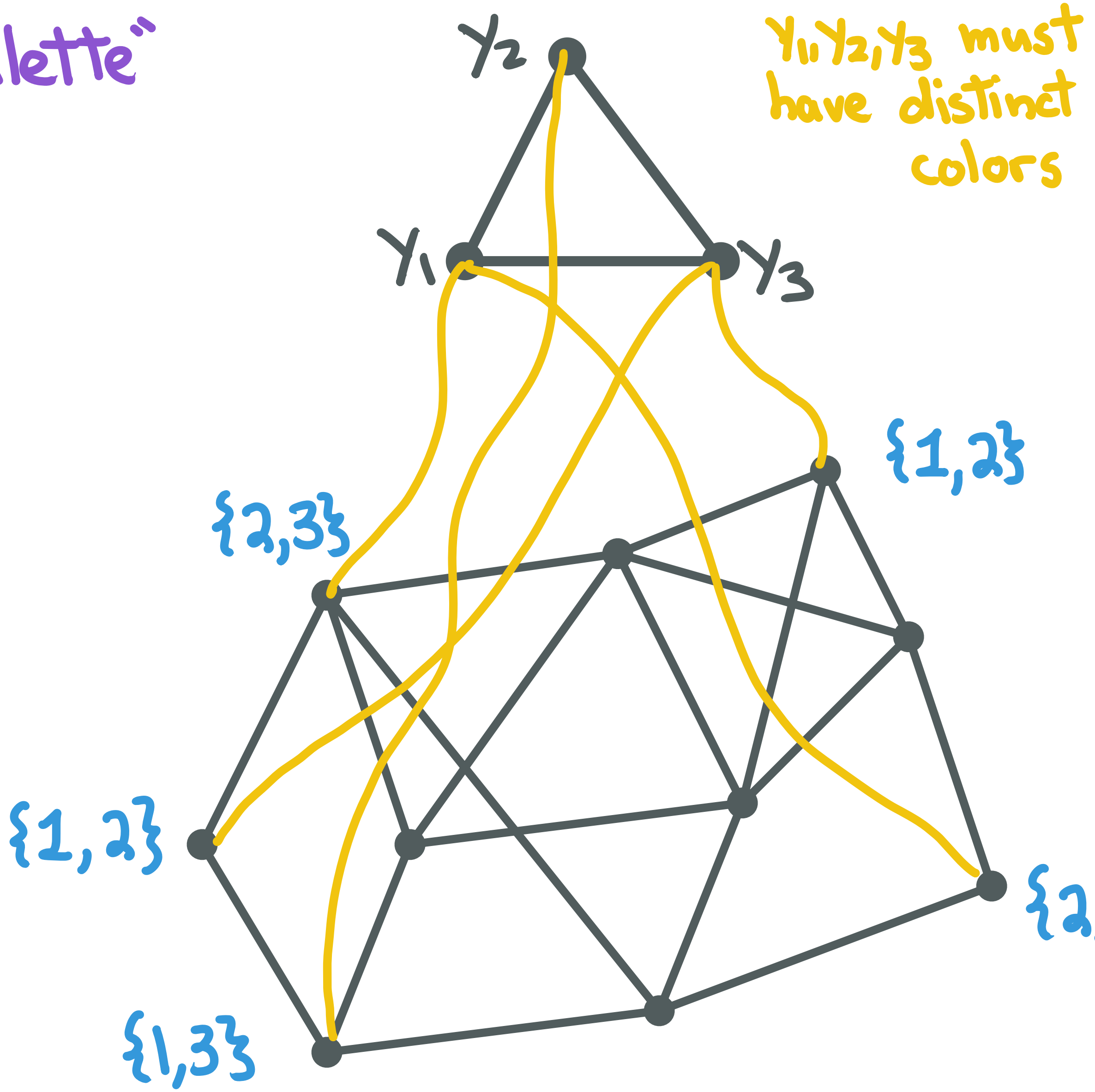
"color palette"



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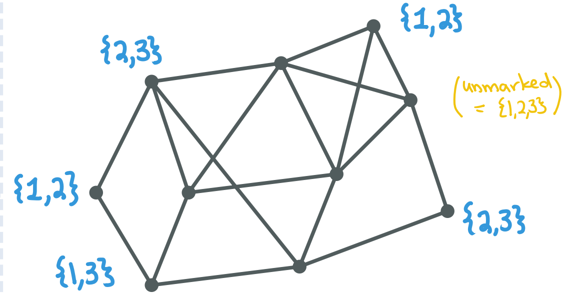

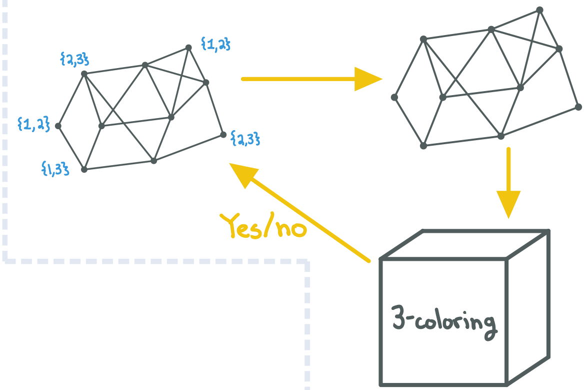

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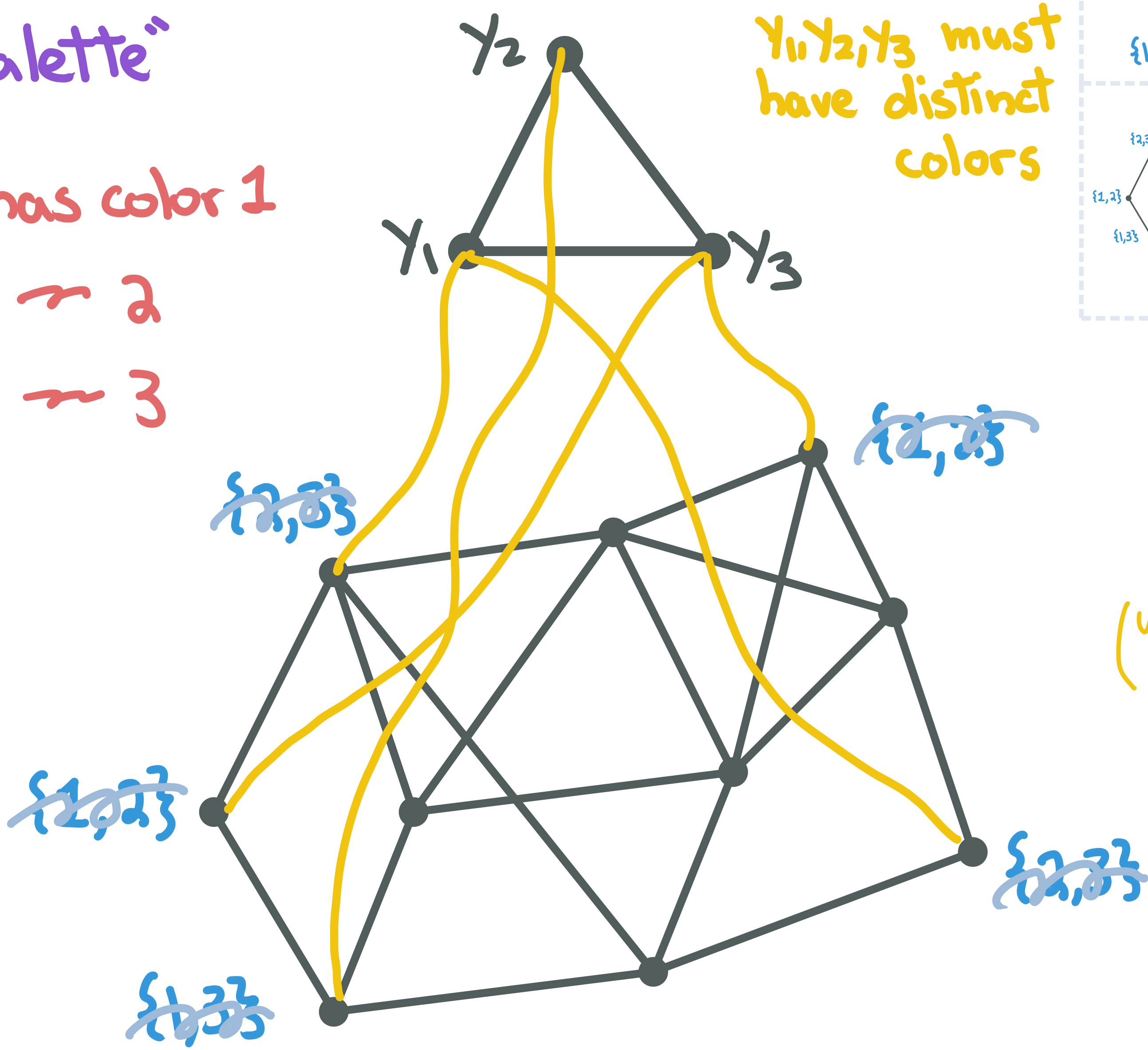
"color palette"

wlog y_1 has color 1

$y_2 \rightsquigarrow 2$

$y_3 \rightsquigarrow 3$



y_1, y_2, y_3 must have distinct colors

(unmarked) = {1,2,3}

Subset 3-coloring

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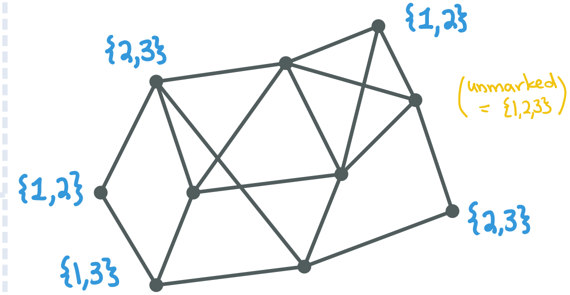
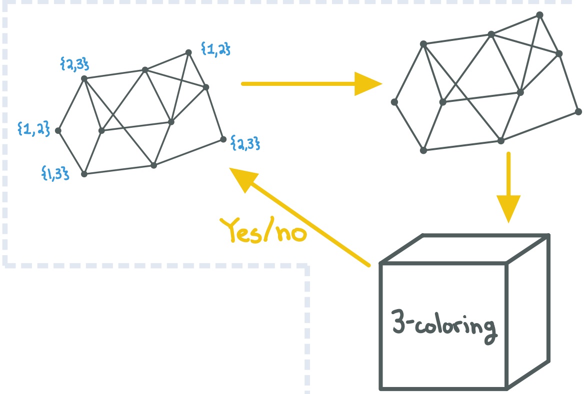
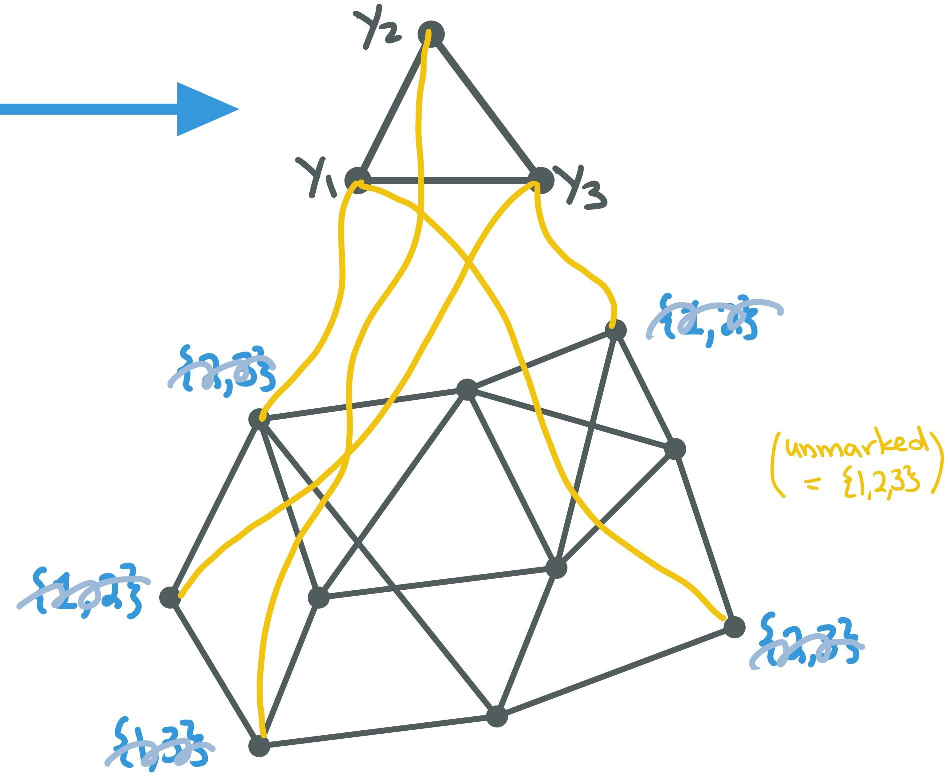
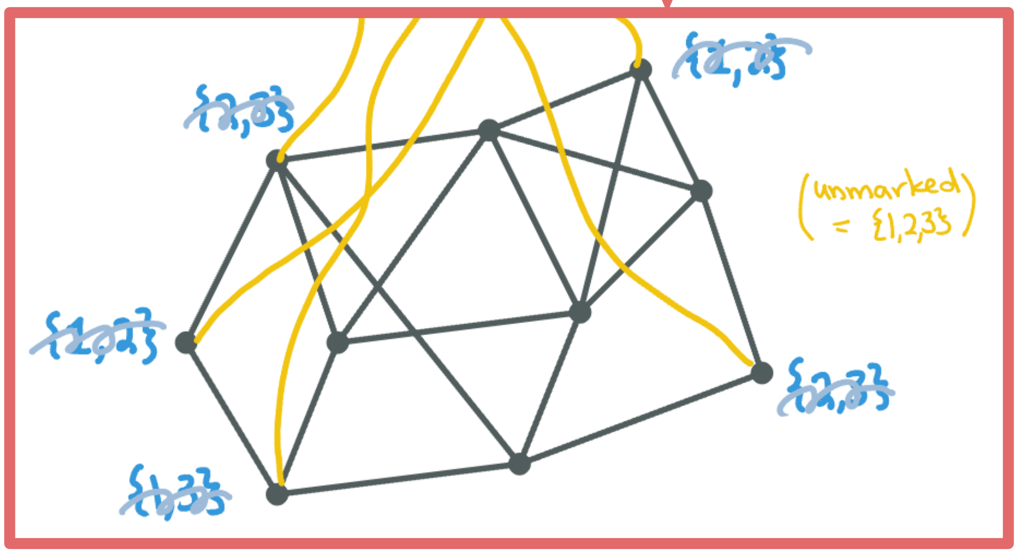




3-coloring on



⇒ subset 3-coloring on



3SAT → "subset 3-coloring" → 3-coloring

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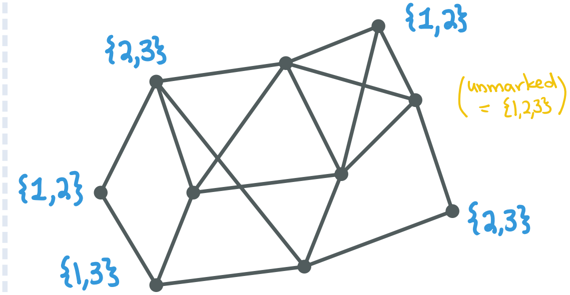

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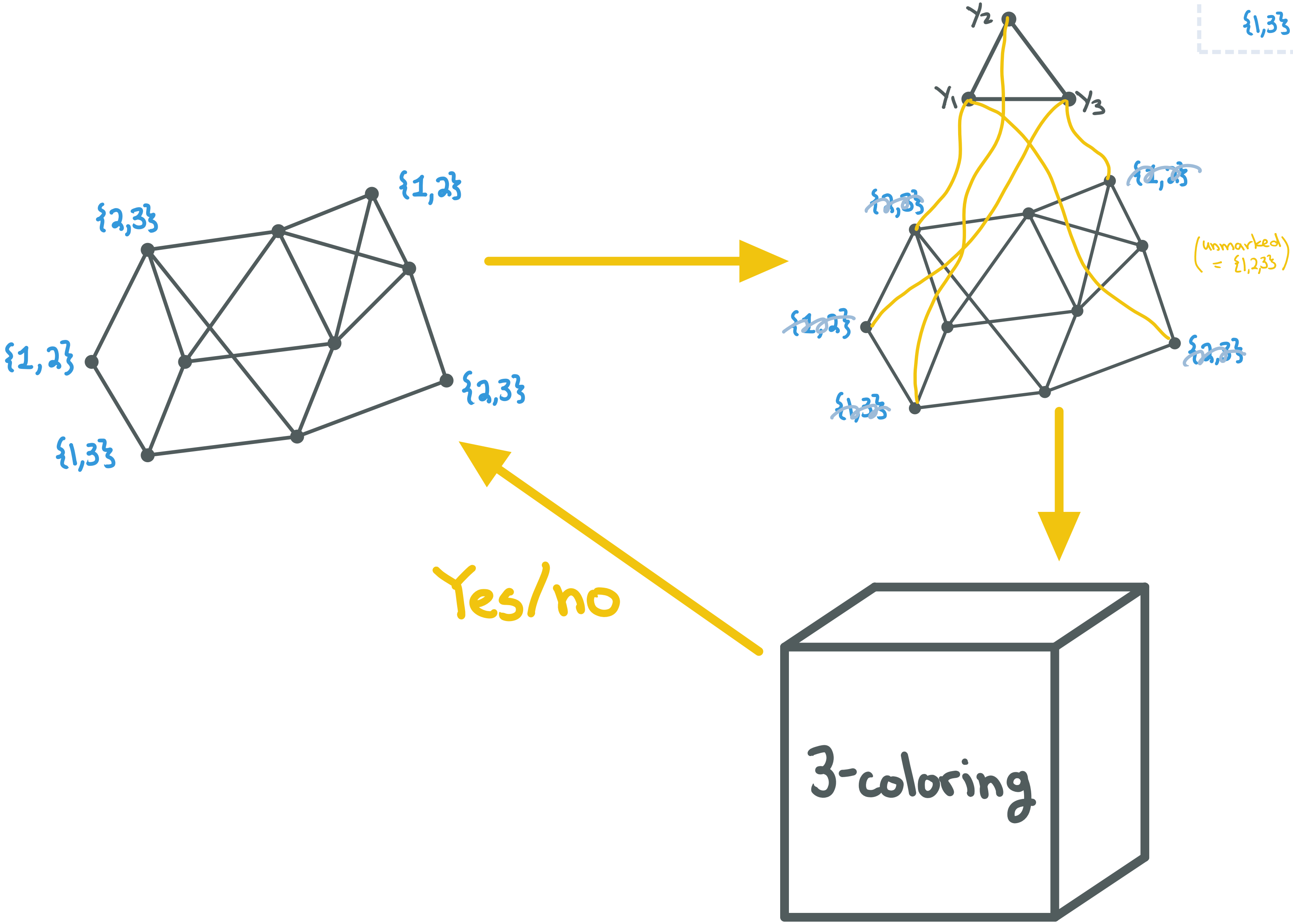

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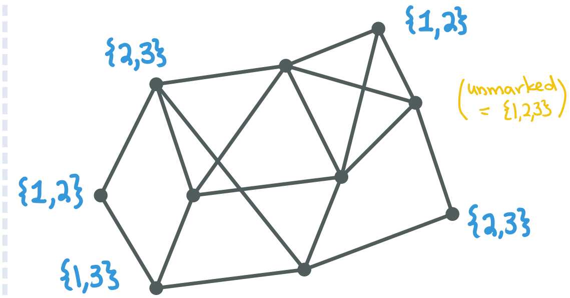

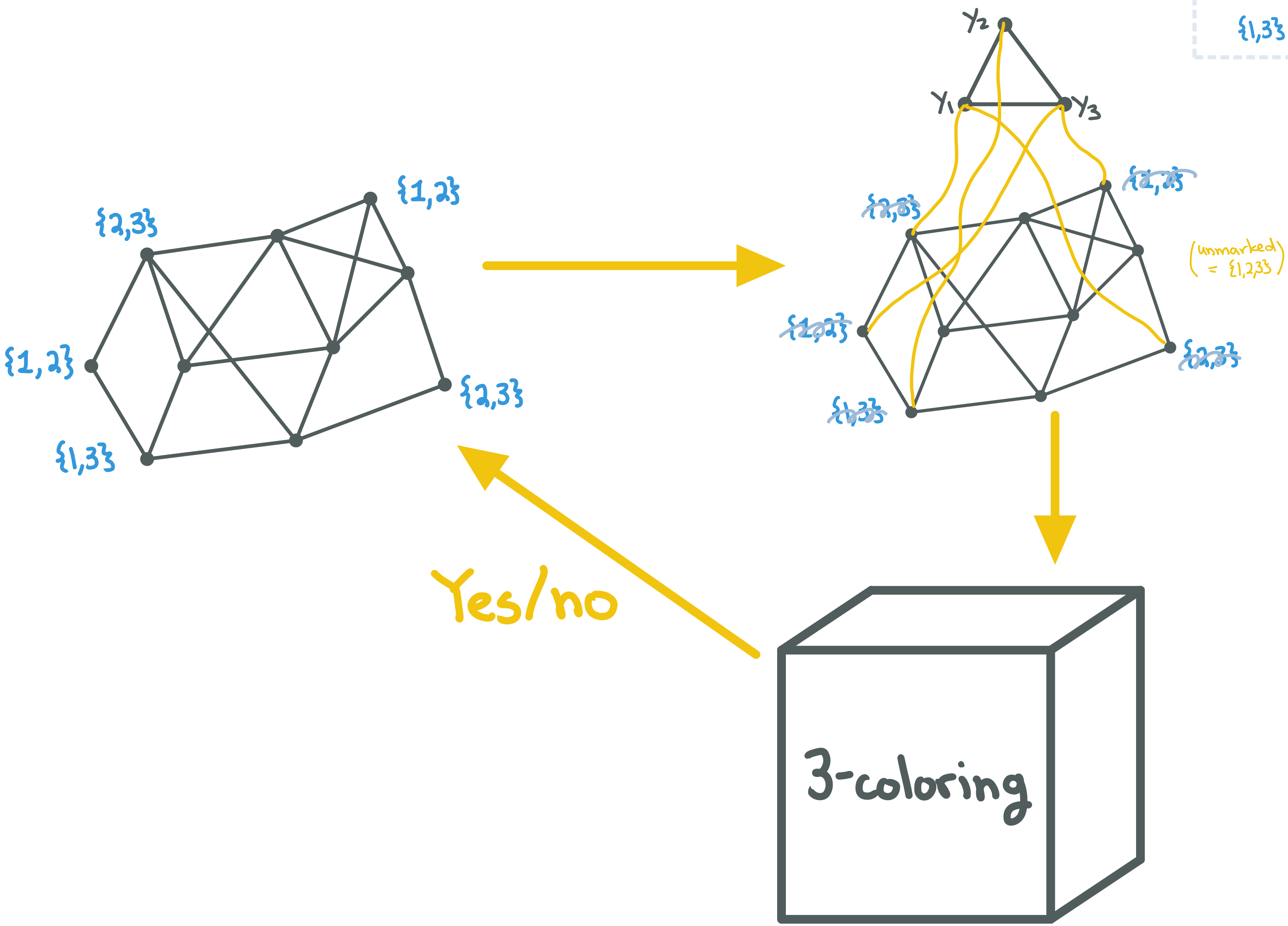

3SAT → "subset 3-coloring" → 3-coloring

claim: polytime algo for subset 3-coloring implies polytime algo for 3-SAT

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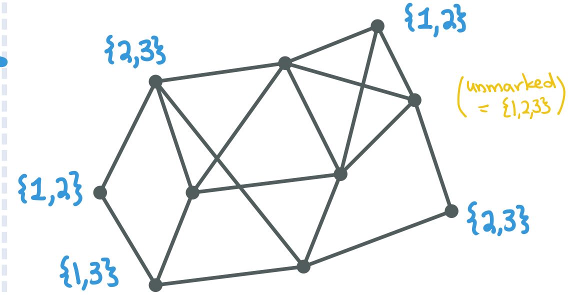

3-SAT

$f(x_1, \ldots, x_n) =$

$(\bar{x}_1 \vee x_2 \vee \bar{x}_3) \wedge$

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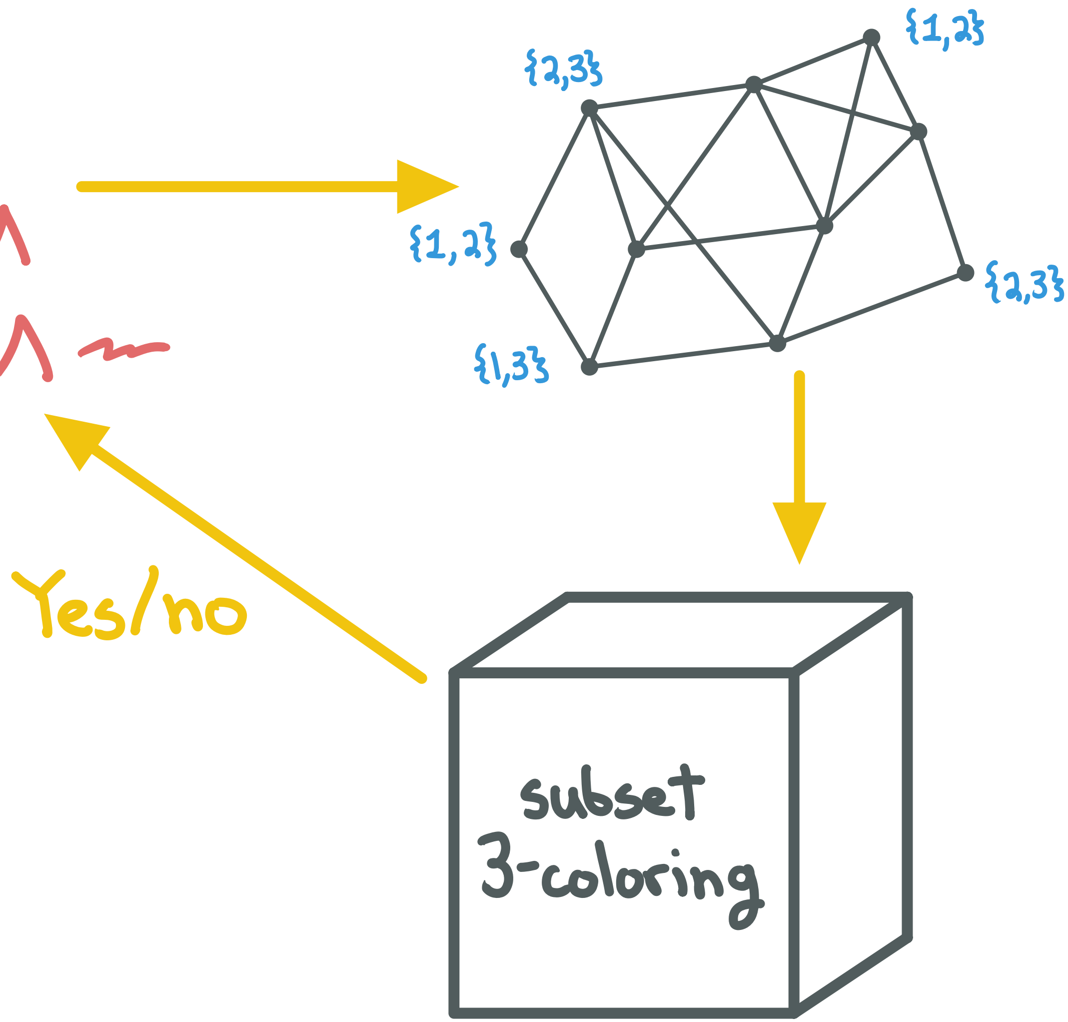

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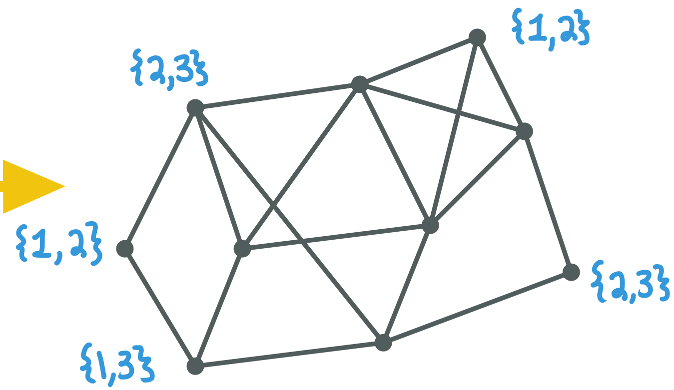

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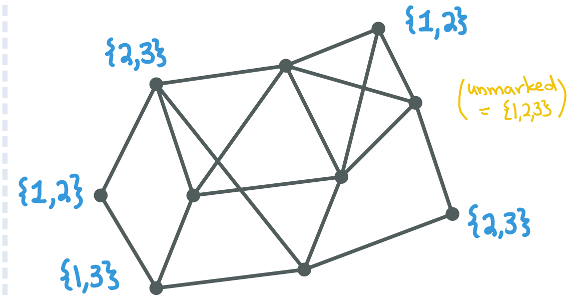

ideas?

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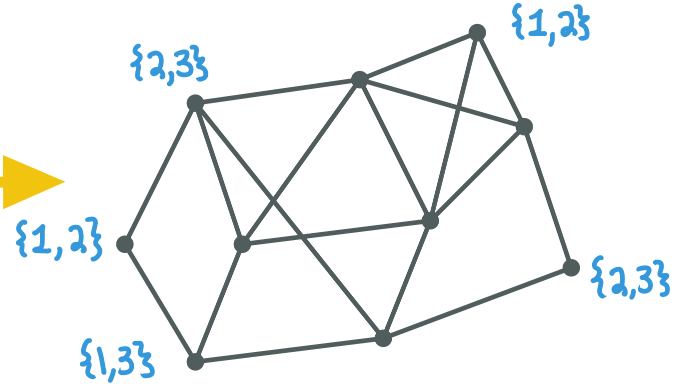

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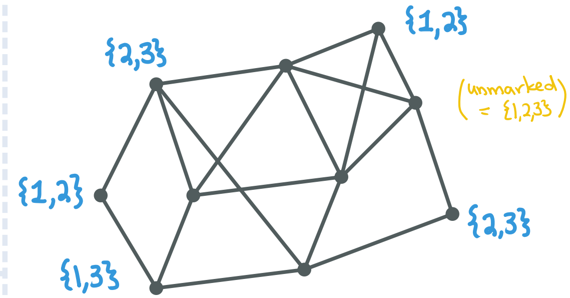

Convenient to name colors:

"T" = "True"

"F" = "False"

"N" = "Neither"

3-SAT

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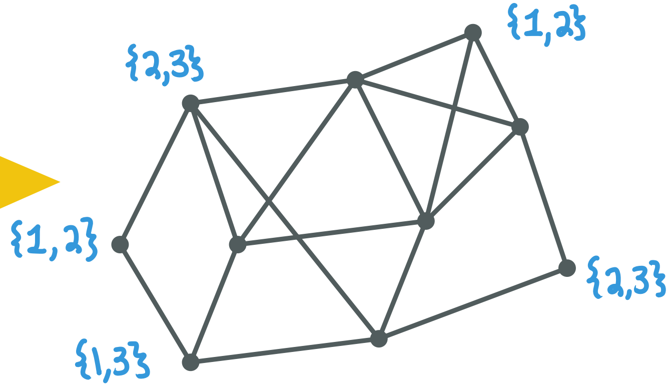

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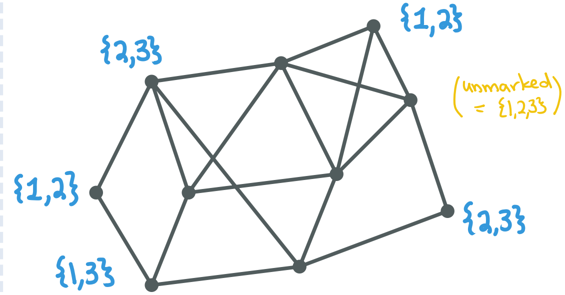


name colors:
"T" = "True"
"F" = "False"
"N" = "Neither"

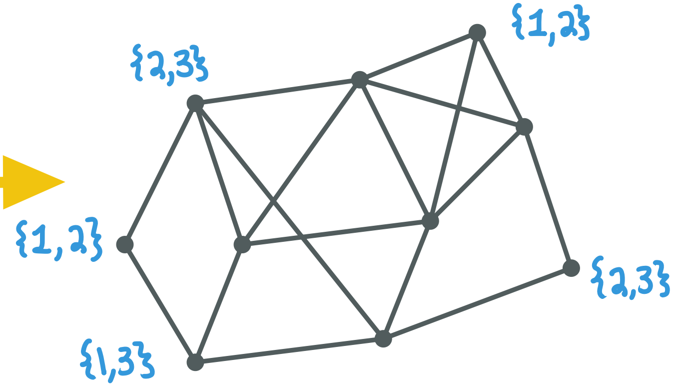
① variables

x_j

interpretation: $x_j = A_j.\text{color}$
$\bar{x}_j = \bar{A}_j.\text{color}$

3-SAT

$f(x_1, \dots, x_n) = (\bar{x}_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3) \wedge \dots$



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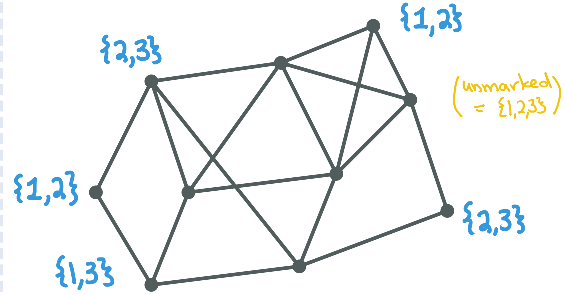

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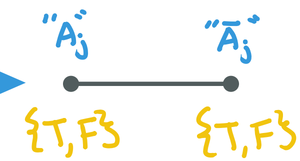
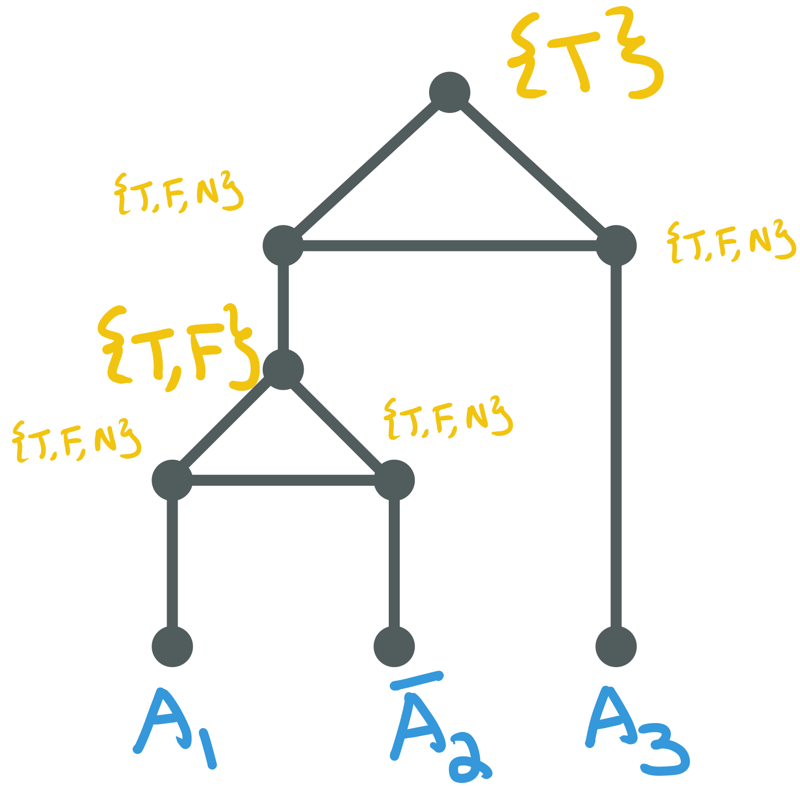
$x_j \rightarrow$

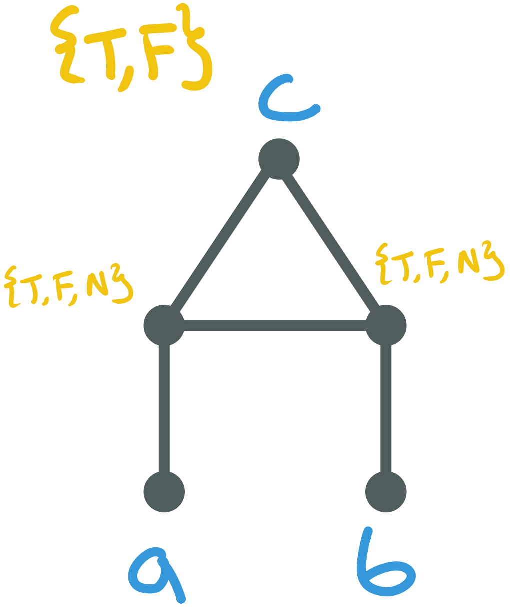


② clauses

$x_1 \vee \bar{x}_2 \vee x_3 \Rightarrow$



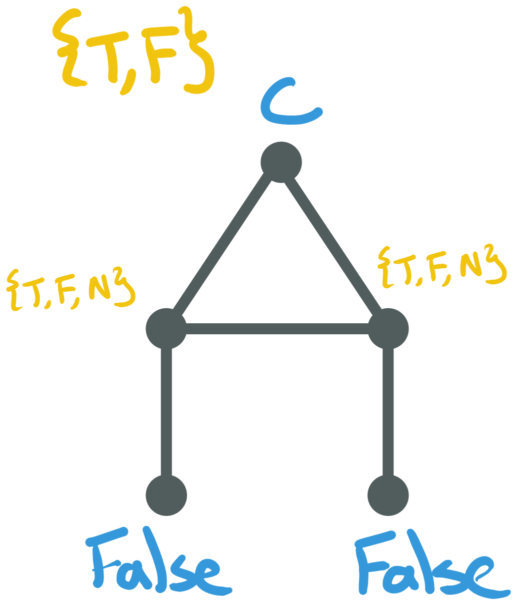
Idea: "or-gadget"



Lemma

c.color = True
possible iff $a \vee b$ = False

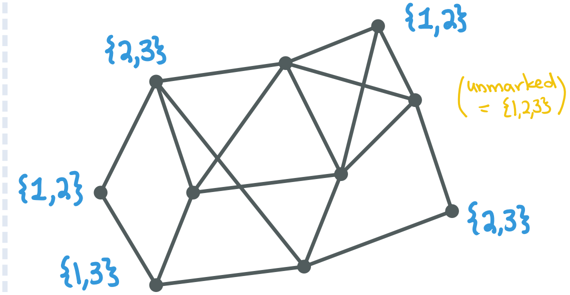
Proof



forces c=False

Subset 3-coloring

for each vertex v, subset $S_v \subseteq \{1,2,3\}$ of colors allowed for v.



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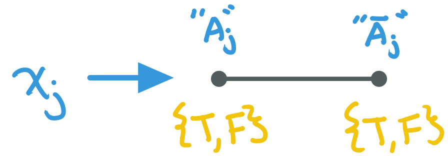

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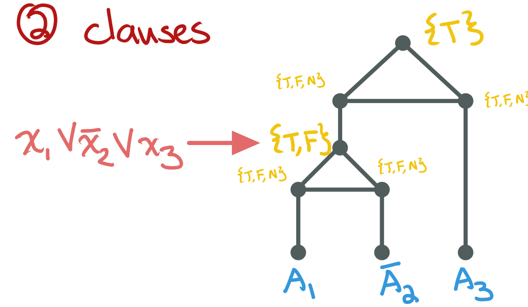

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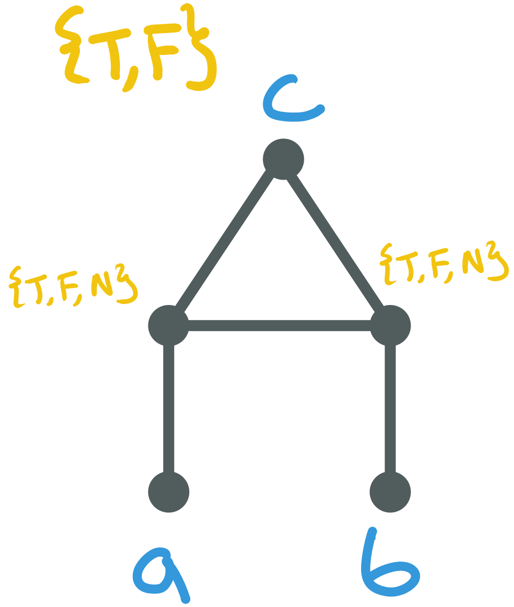

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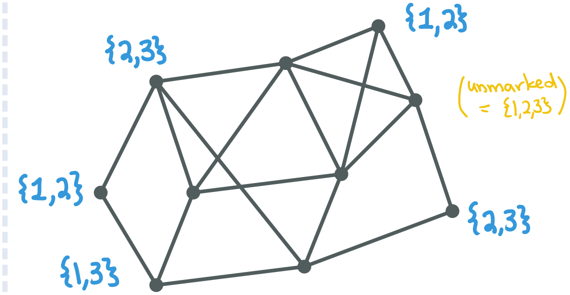

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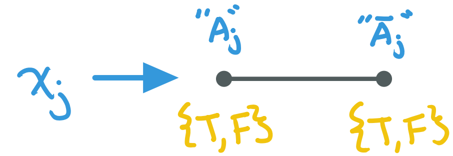

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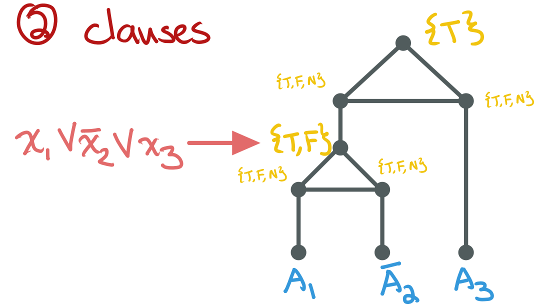
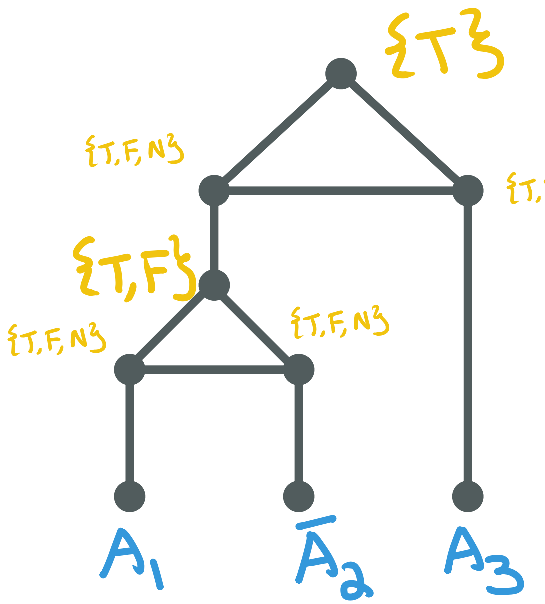


$x_1 \vee \bar{x}_2 \vee x_3 \Rightarrow$

combines 2 ∨-gadgets



3SAT → "subset 3-coloring" → 3-coloring

claim: polytime algo for subset 3-coloring implies polytime algo for 3-SAT

Subset 3-coloring

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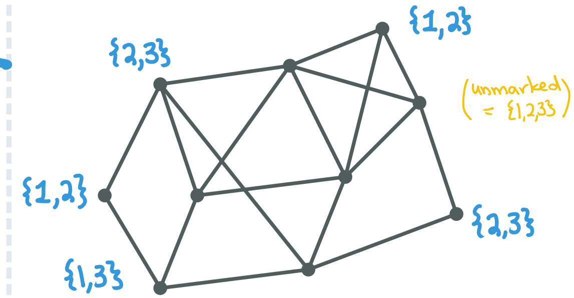
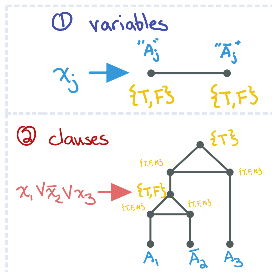
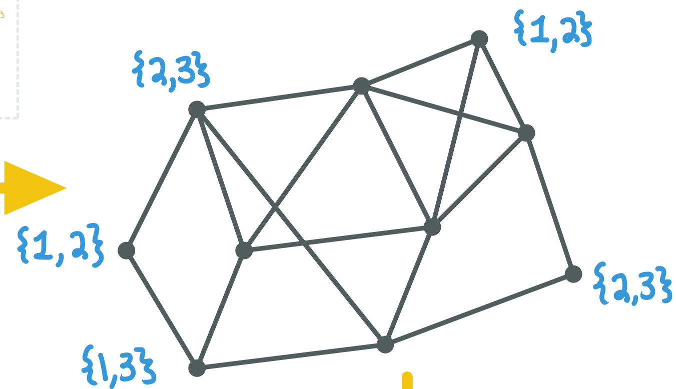
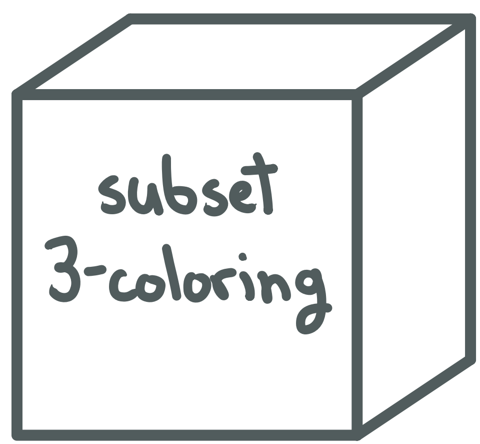




3-SAT

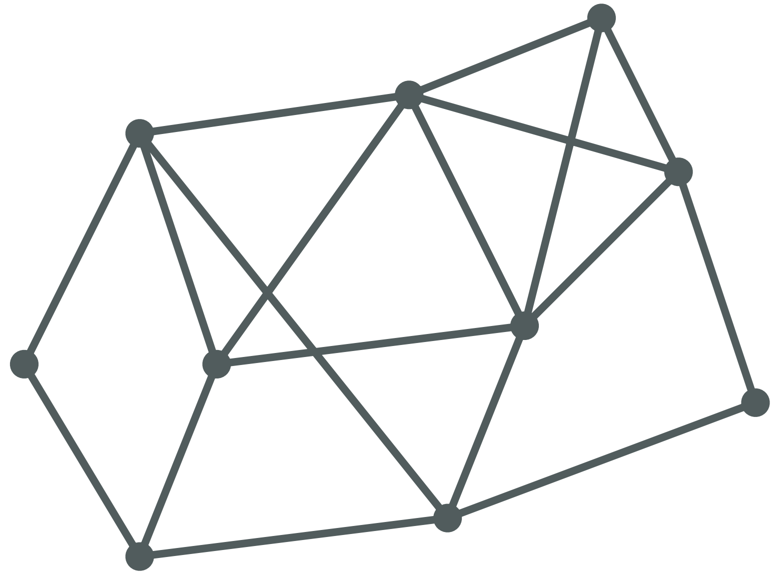
$$f(x_1, \ldots, x_n) = (\bar{x}_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3) \wedge \ldots$$





Yes/no

Dominating Set



$S \subseteq V$ is **dominating** if all $v \in V$ either

(a) is in S (b) has a neighbor in S

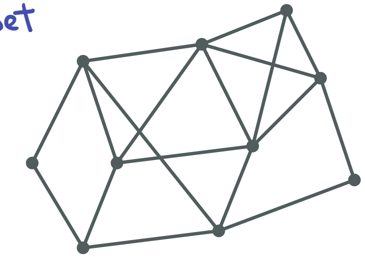
Goal: min cardinality dominating set

Decision version: dom. set w/ $\leq k$ vertices?

Theorem Dom set is NP-Hard ← Ideas?

claim: polytime algo for dominating set implies polytime algo for 3-SAT

Dominating Set



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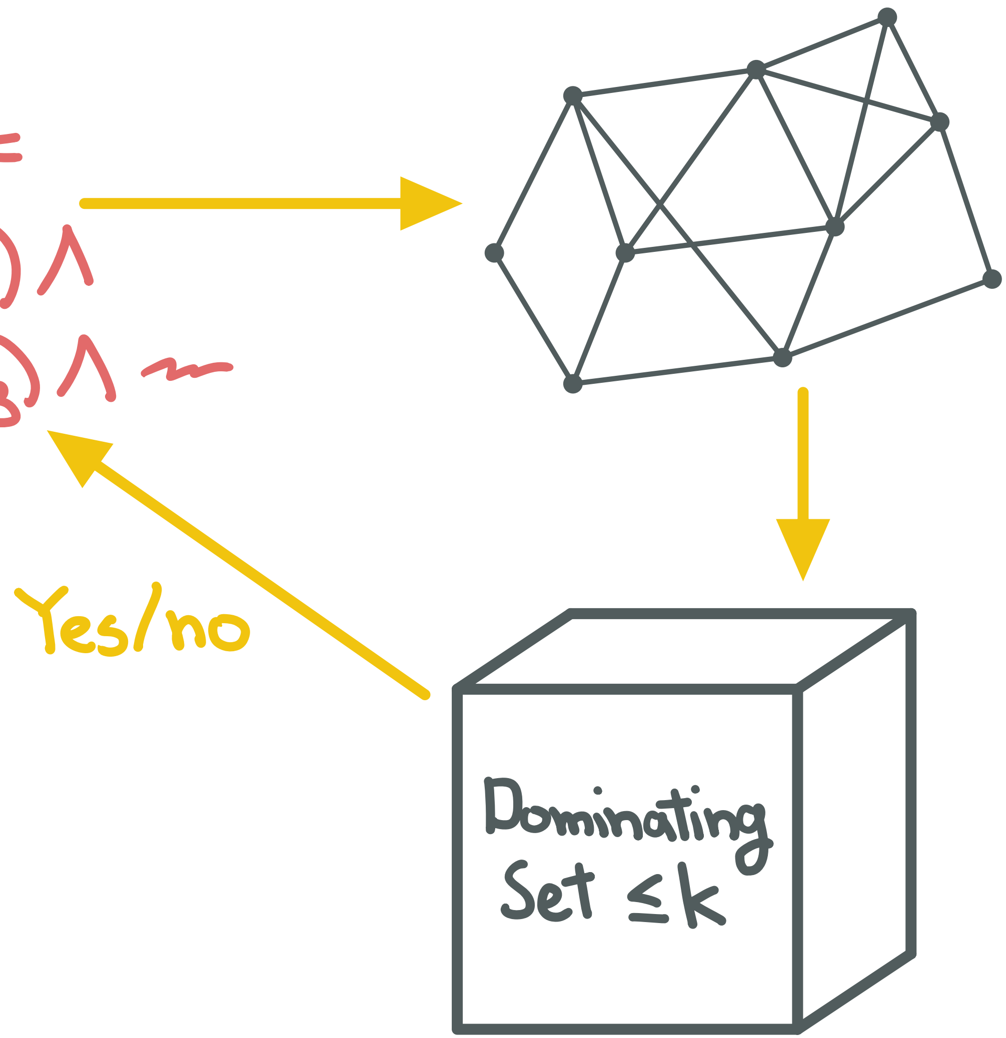
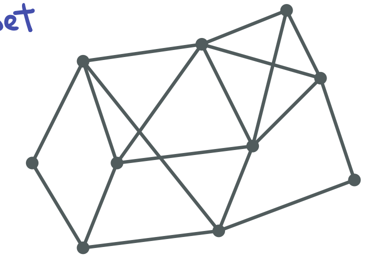


claim: polytime algo for dominating set implies polytime algo for 3-SAT

Ideas?

Dominating Set



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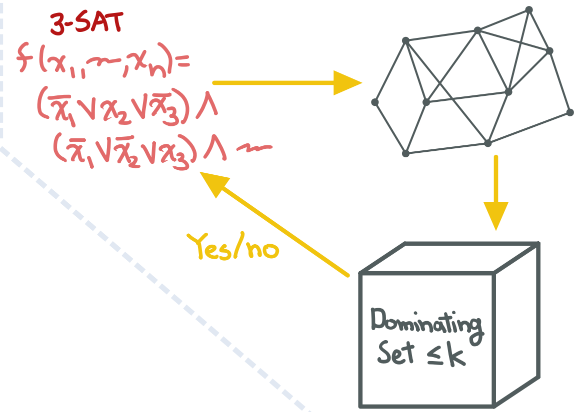

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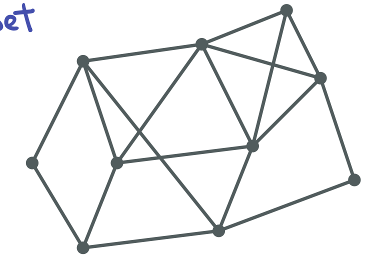
Dominating Set



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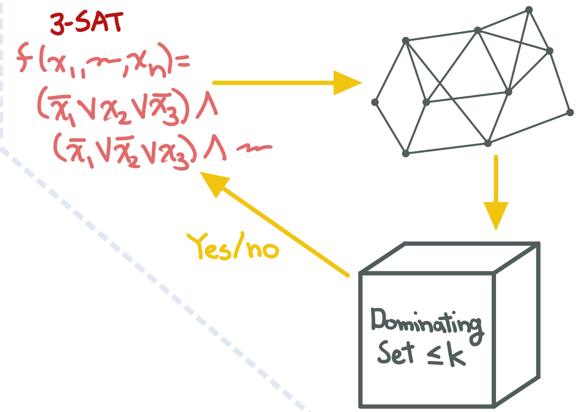
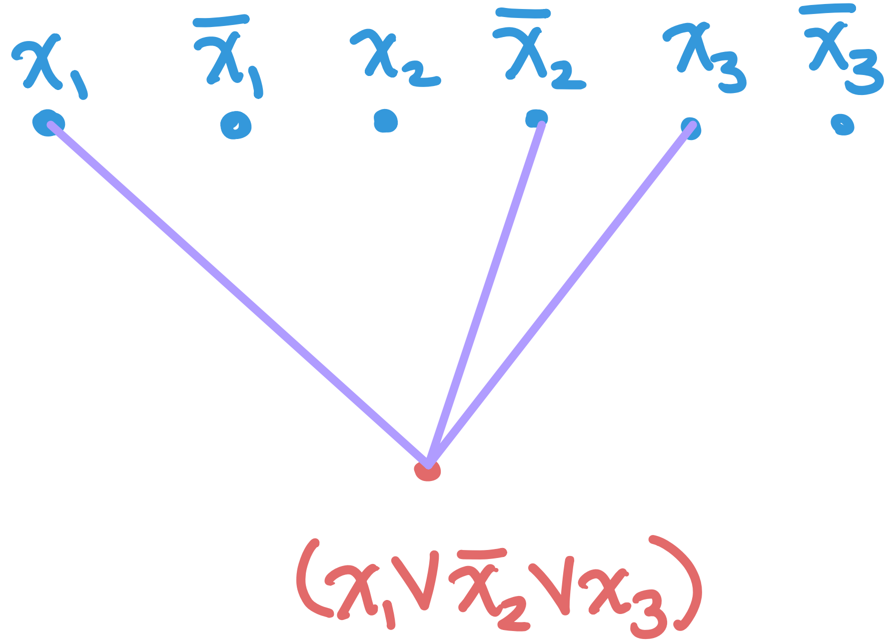


literals = vertices

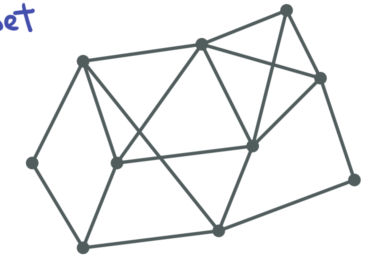
clauses = vertices



claim: polytime algo for dominating set implies polytime algo for 3-SAT

$$(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})$$

Dominating Set



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