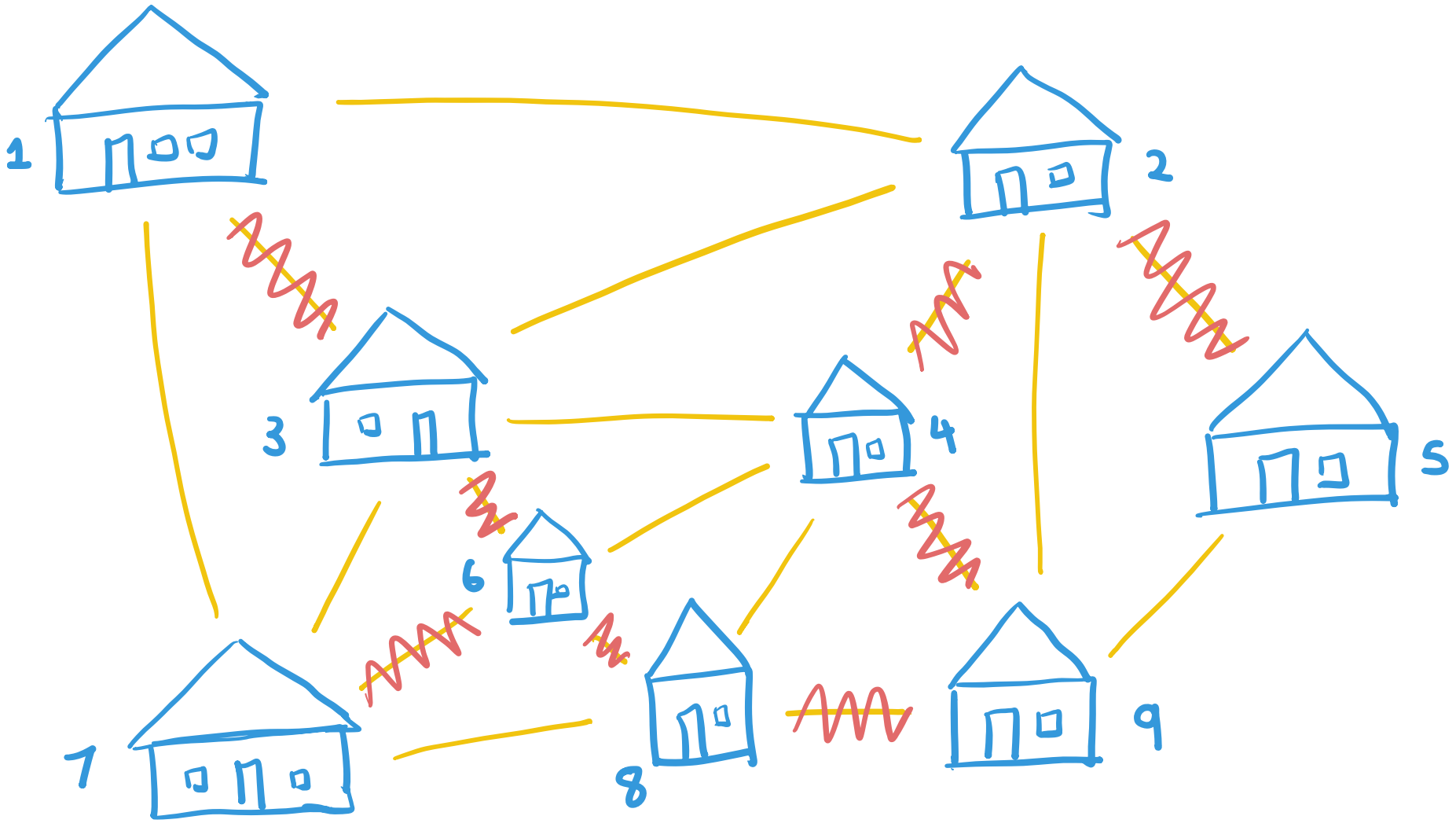
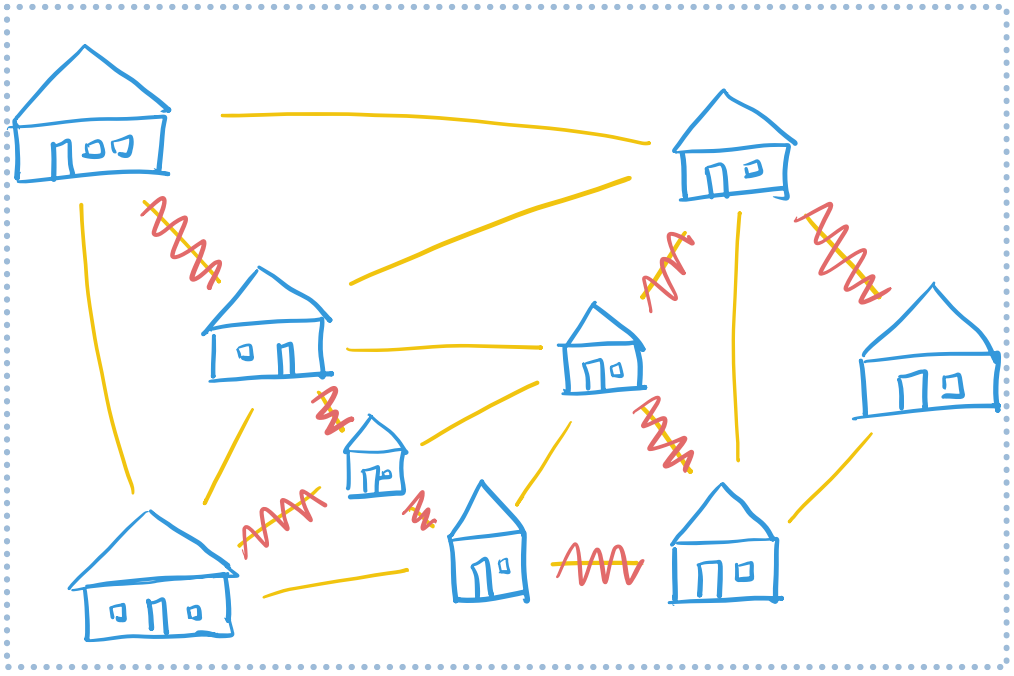
Spanning Trees



GOAL: connect town w/ minimum amount of electrical wire

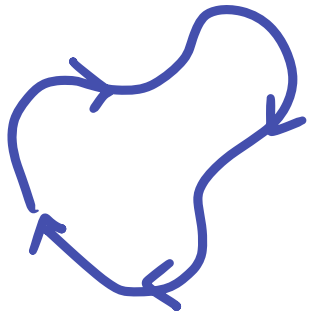
Quick observations

- no reason to have cycles
- biased towards short wires
- unused wires have alternative paths (which may be longer)

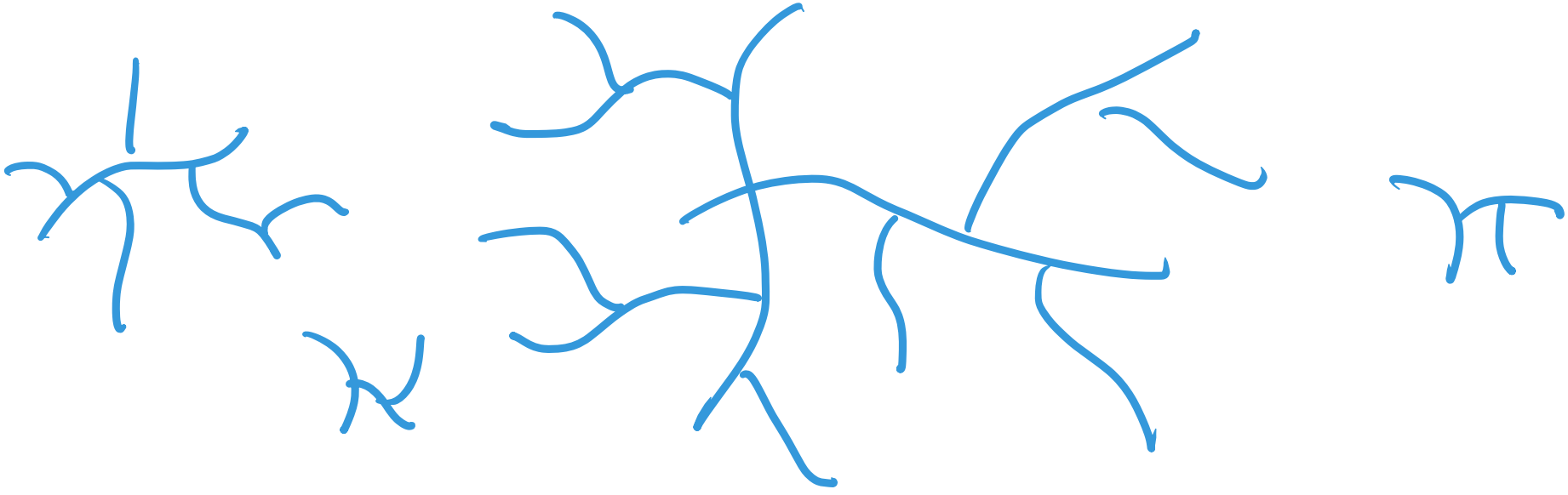


$G=(V,E)$ undirected, $F \subseteq E$ subset of edges.

"circuit" (nonempty) walk that starts/ends at same vertex



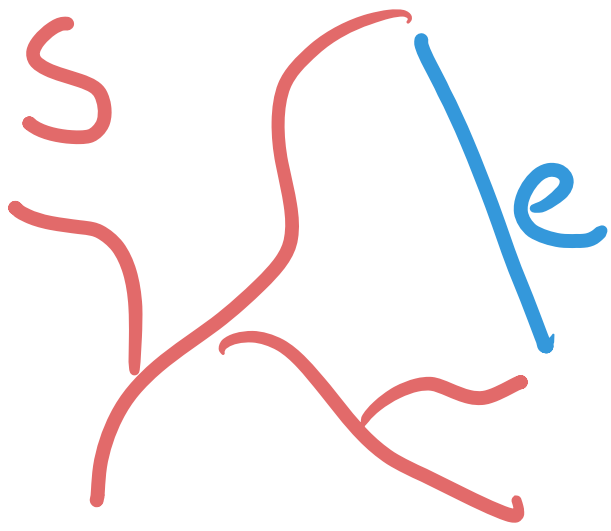
"forest": set of edges w/ no circuits



"Tree": forest w/ one connected component

$G=(V,E)$ undirected, $S \subseteq E$

"S spans e" if endpoints of e are connected by S



$\text{span}(S) \overset{\text{def}}{=} \{e \in E : S \text{ spans } e\}$

"S is spanning" if S spans every $e \in E$

2 problems

1. compute a forest $F \subseteq E$ of **maximum** cardinality (or weight)

2. compute a spanning set $S \subseteq E$ of minimum cardinality (or weight)

"circuit": (nonempty) walk that starts/ends at same vertex

"forest": set of edges w/ no circuits

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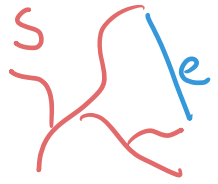

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"S is spanning" if S spans every $e \in E$

let $G=(V,E)$ be connected

Lemma. $F \subseteq E$ forest, $S \subseteq E$ spanning. Then

$$|F| \leq n-1 \leq |S|$$

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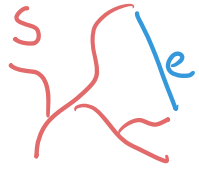

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"S is spanning" if S spans every $e \in E$

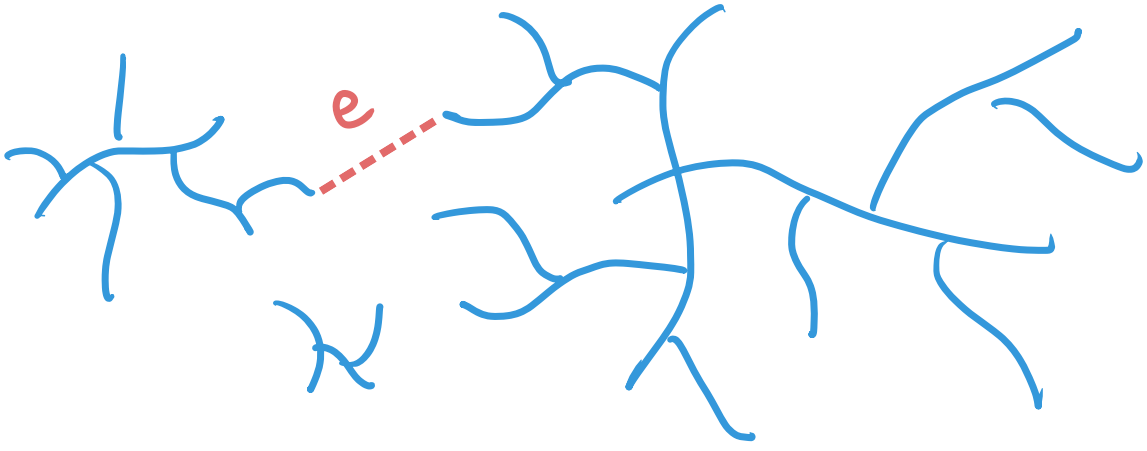
$G=(V,E)$ connected

Lemma. $F \subseteq E$ forest, $S \subseteq E$ spanning. Then

$$|F| \leq n-1 \leq |S|$$

Proof. let $F=\emptyset$ be an empty forest

add edges to F one at a time



each edge decreases # connected components by 1, from n down to 1

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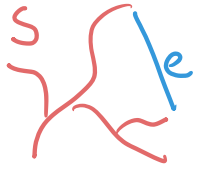

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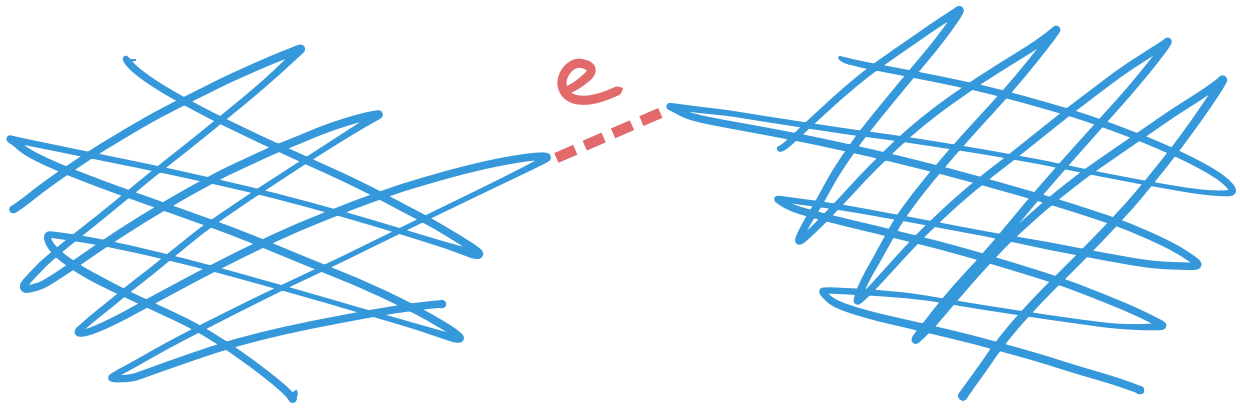
$G=(V,E)$ connected

Lemma. $F \subseteq E$ forest, $S \subseteq E$ spanning. Then

$$|F| \leq n-1 \leq |S|$$

Proof. let S be a spanning set.

there is 1 connected component w/r/t S



if we delete an edge, # conn. components increase by at most 1.

deleting all edges $\Rightarrow$ n conn. components

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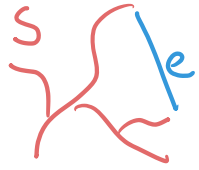

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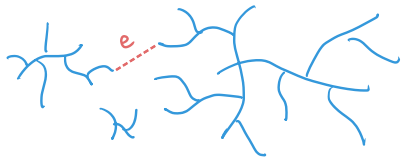
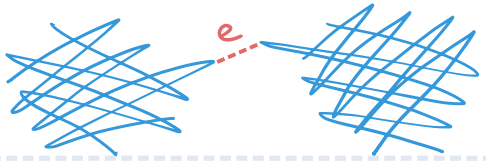


$|F| \leq n-1 \leq |S|$



Now suppose G has k connected components.

claim: $|F| \leq n-k \leq |S|$

why?

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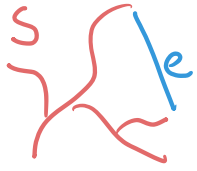

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"S is spanning" if S spans every $e \in E$

$F \subseteq E$ forest

F is a **maximal forest** if for all $e \in E \setminus F$,
$F + e$ is not a forest.

F is a **maximum forest** if it has maximum size over all forests.

$S \subseteq E$ spanning.

S is a **minimal spanning set** if for all $e \in S$,
$S - e$ is not a spanning set

S is a **minimum spanning set** if it has minimum size over all spanning sets.

2 problems

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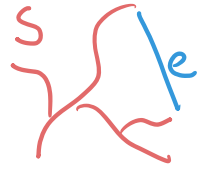

$\text{span}(S) \overset{\text{def}}{=} \{e \in E : S \text{ spans } e\}$

"S is spanning" if S spans every $e \in E$

G has k conn. components.
$F \subseteq E$ forest, $S \subseteq E$ spanning.
then $|F| \leq n-k \leq |S|$

Theorem. $F \subseteq E$ forest, $S \subseteq E$ spanning.

1. If F is *maximal* forest, then F is spanning.
2. If S is *minimal* spanning set, then S is a forest.

Corollary.

- Maximal forests are (globally) maximum forests
- Minimal spanning sets are (globally) minimum spanning sets.

In both cases they are spanning forests w/ $n-k$ edges

2 problems

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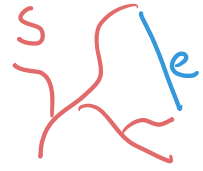

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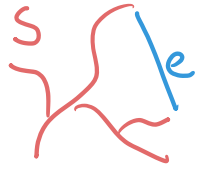

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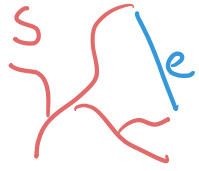

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Corollary.

- Maximal forests are maximum (globally) forests
- Minimal spanning sets are minimum (globally) spanning sets.

In both cases they are spanning forest w/ $n-k$ edges

Corollary. If G is connected, there is a spanning tree w/ $n-1$ edges.

2 problems

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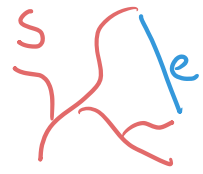

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G has k conn. components.
$F \subseteq E$ forest, $S \subseteq E$ spanning.
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2 problems

1. compute a forest $F \subseteq E$ of **maximum** cardinality (or **weight**)

2. compute a spanning set $S \subseteq E$ of **minimum** cardinality (or **weight**)

Assume that all edge weights are distinct

"circuit": (nonempty) walk that starts/ends at same vertex

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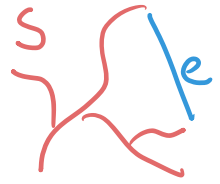

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Good edges

2 problems

1. compute a forest $F \subseteq E$ of maximum cardinality (or weight)
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$e \in E$ is "good" if

For every set of edges $F \subseteq E - e$ that spans e (and doesn't include e),

$$w(e) < \max_{f \in F} w(f).$$

Good edges

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Lemma 13.5. *Let $e \in E$. Then the following conditions are equivalent.*

1. *Every minimum weight spanning tree contains e.*
2. *For every set of edges $F \subseteq E - e$ that spans e (and doesn't include e),* (e is good)

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Proof. Assume #1.

2 problems

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Proof. Assume #2.

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generic-MST$(G = (V, E), w : E \to \mathbb{R})$

1. $T \leftarrow \emptyset$
2. Until T is a spanning tree.
   A. Add a good edge $e \in E \setminus \text{span}(T)$ to T
   // Or throw an error if no such edge exists.
3. Return T.

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Q: how to identify good edges?
are there always good edges?

Lemma. $H \subseteq E$

$e = \arg\min_e \{w(e) : e \in E \setminus \text{span}(H)\}$ (min weight unspanned edge)

then e is good

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"Greedy algorithm"

`greedy-MST`$(G = (V, E), w : E \to \mathbb{R})$

/ Also known as Kruskal's algorithm. */*

1. Sort and index $E = \{e_1, \ldots, e_n\}$ in increasing order of weight.
2. Set $T \leftarrow \emptyset$.
3. For $i = 1, \ldots, n$:

   / Below we discuss how to implement the following predicate efficiently. */*

   A. If $T + e_i$ is a forest:

      / e_i is the minimum weight edge not spanned by T */*

      1. Set $T \leftarrow T + e_i$.
4. Return T.

why is e_i good?

2 problems

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Lemma. $H \subseteq E$

$e = \arg\min_{e} \{w(e) : e \in E \setminus \text{span}(H)\}$ (min weight unspanned edge)

then e is good

"Parallel algorithm"

parallel-MST($G = (V, E), w : E \to \mathbb{R}$)

/* *Also known as Borůvka's algorithm.* */

1. $T \leftarrow \emptyset$
2. While T more than 1 connected component:
   A. For each component S of T, identify the smallest weight edge in $\partial(S)$.
   B. Add all of these edges to T.
3. Return T

2 problems

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Lemma. $H \subseteq E$

$e = \arg\min_e \{w(e) : e \in E \setminus \text{span}(H)\}$ (min weight unspanned edge)

then e is good

"Search algorithm"

search-MST$(G = (V, E), w : E \to \mathbb{R})$

/* *Also known as Prim's algorithm.* */

1. Set $T \leftarrow \emptyset$ and $S \leftarrow \{v\}$ for an arbitrary vertex v.

/* *We maintain the invariant that S is the connected component of v in T.* */

2. While T is not a spanning tree:

   /* *We can accelerate the following steps with a Fibonacci heap.* */

   A. Let e be the minimum weight edge in $\partial(S)$.

   B. Set $T \leftarrow T + e$, $S \leftarrow S \cup e$.

3. Return T.

2 problems

1. compute a forest $F \subseteq E$ of **maximum** cardinality (or weight)
2. compute a spanning set $S \subseteq E$ of minimum cardinality (or weight)

$e \in E$ is "good" if

For every set of edges $F \subseteq E - e$ that spans e (and doesn't include e),

$$w(e) < \max_{f \in F} w(f).$$

Lemma 13.5. *Let $e \in E$. Then the following conditions are equivalent.*

1. *Every minimum weight spanning tree contains e.*
2. *For every set of edges $F \subseteq E - e$ that spans e (and doesn't include e),* (e is good)

$$w(e) < \max_{f \in F} w(f).$$

generic-MST$(G = (V, E), w : E \to \mathbb{R})$

1. $T \leftarrow \emptyset$
2. Until T is a spanning tree.
   A. Add a good edge $e \in E \setminus \text{span}(T)$ to T
   // *Or throw an error if no such edge exists.*
3. Return T.

Running times

generic-MST($G = (V, E), w : E \to \mathbb{R}$)

1. $T \leftarrow \emptyset$
2. Until T is a spanning tree.
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greedy-MST($G = (V, E), w : E \to \mathbb{R}$)

/ Also known as Kruskal's algorithm. */*

1. Sort and index $E = \{e_1, \ldots, e_n\}$ in increasing order of weight.
2. Set $T \leftarrow \emptyset$.
3. For $i = 1, \ldots, n$:
   / Below we discuss how to implement the following predicate efficiently. */*
   A. If $T + e_i$ is a forest:
      / e_i is the minimum weight edge not spanned by T */*
      1. Set $T \leftarrow T + e_i$.
4. Return T.

parallel-MST($G = (V, E), w : E \to \mathbb{R}$)

/ Also known as Borůvka's algorithm. */*

1. $T \leftarrow \emptyset$
2. While T more than 1 connected component:
   A. For each component S of T, identify the smallest weight edge in $\partial(S)$.
   B. Add all of these edges to T.
3. Return T

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/ Also known as Prim's algorithm. */*

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Greedy algorithm

greedy-MST$(G = (V, E), w : E \to \mathbb{R})$

/ Also known as Kruskal's algorithm. */*

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   A. If $T + e_i$ is a forest:

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Disjoint union data structure

1. union(u, v): Given two elements u and v, take the union of the their sets.
2. same-set(u, v): Returns whether or not x and y are in the same set.
3. new-set(x): Given a new element x, creates the singleton set $\{x\}$.

Theorem 13.8. *There is a data structure that can implement the* union, same-set, *and* new-set *operations listed above that, for any m operations over n total elements, takes $O(m + n\alpha(n))$ time, where $\alpha(n)$ is the inverse Ackerman function.*

"Parallel algorithm"

parallel-MST($G = (V, E), w : E \to \mathbb{R}$)

/* *Also known as Borůvka's algorithm.* */

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Priority queues / heaps

1. `insert`(k, p): inserts an key k with priority p.
2. `decrease`(k, p): Let k be a key in the heap. If p is less than the current priority of k, decrease the priority to p.
3. `remove-min()`: removes and returns the key value pair (k, p) with the minimum priority p.

Fact 8.1. *There exists a data structure, called a* **Fibonacci heap**, *that has the following guarantee. Over any sequence of I calls to* `insert-key`, D *calls to* `decrease-key`, *and R calls to* `remove-min`, *the Fibonacci heap takes*

$$O(I + D + R \log n)$$

time in total.

(same as Dijkstra's)

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O(1) 1. insert(k, p): inserts an key k with priority p.

O(1) 2. decrease(k, p): Let k be a key in the heap. If p is less than the current priority of k, decrease the priority to p.

O(log n) 3. remove-min(): removes and returns the key value pair (k, p) with the minimum priority p.

Running times

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$O(m \log n)$

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$O(m \log n)$

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   B. Set $T \leftarrow T + e$, $S \leftarrow S \cup e$.
3. Return T.

$O(m + n \log n)$