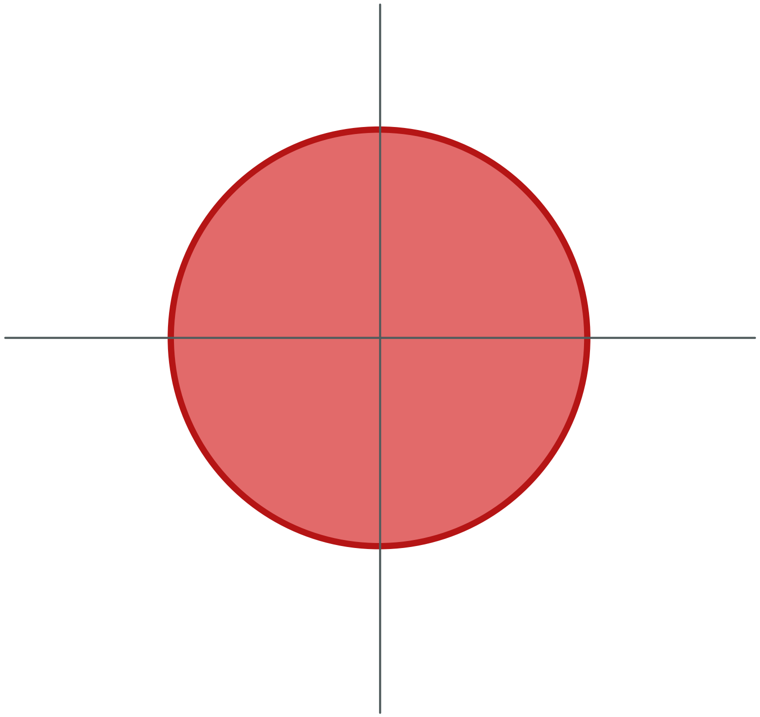
DNF counting

How to estimate the area of a circle?



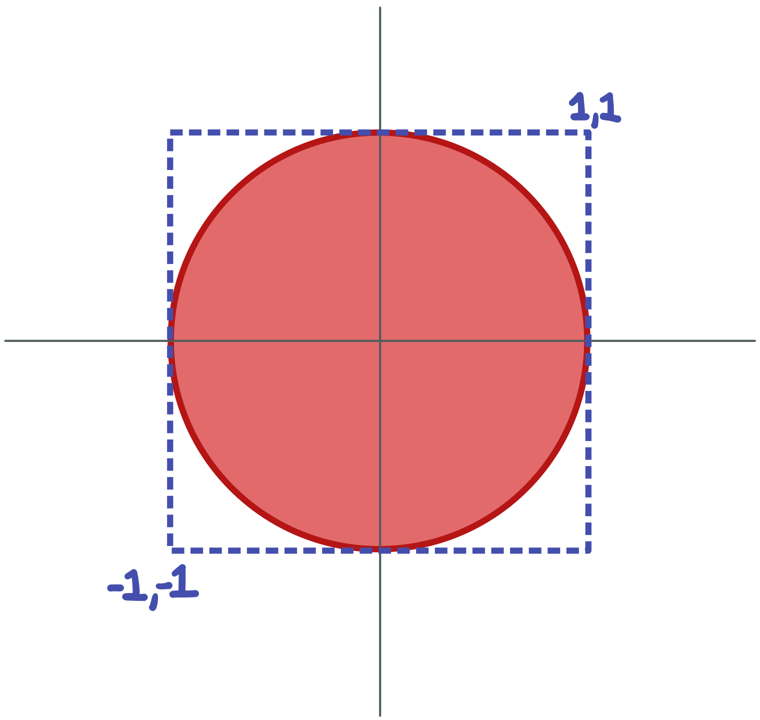
How to estimate the area of a circle?

Multiplicative Chernoff bound

let $X_1, \dots, X_n \in [0,1]$ be independent

$\mu = E[X_1 + \dots + X_n]$ $\quad \varepsilon \in (0,1)$

- $P[X_1 + \dots + X_n \ge (1+\varepsilon)\mu] \le e^{-\varepsilon^2 \mu / 3}$
- $P[X_1 + \dots + X_n \le (1-\varepsilon)\mu] \le e^{-\varepsilon^2 \mu / 2}$



sample $p \in [-1,1]^2$ unif. random

$$P[p \in \bullet] = \frac{\text{Area}(\bullet)}{4} \quad (\approx .785\dots)$$

$X_1, \dots, X_k \in \{0,1\}$

$$X_i = \begin{cases} 1 & \text{if } i\text{th sample in } \bullet \\ 0 & \text{otherwise} \end{cases}$$

$$P[X_1 + \dots + X_k \ge (1+\varepsilon) P[p \in \bullet] k] \le e^{-\Omega(\varepsilon^2 P[p \in \bullet] k)}$$

$$\le \delta/2 \text{ for } k \ge O\left(\frac{\log(1/\delta)}{P[p \in \bullet]\,\varepsilon^2}\right)$$

similar for $P[X_1 + \dots + X_k \le (1-\varepsilon) P[p \in \bullet] k]$

(disjunctive normal form)

DNF $\varphi(x_1, \dots, x_n) = (x_1 \wedge \bar{x}_3 \wedge x_5 \wedge \dots) \vee (x_3 \wedge \bar{x}_4 \wedge \dots) \vee \dots$

easy to decide if φ satisfiable...

How many assignments satisfy φ?

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Can we estimate this? $\longrightarrow$ $P[\varphi(x) = \text{TRUE}] = \left(\begin{array}{c}\#\text{ satisfying} \\ \text{assignments}\end{array}\right) / 2^n$

$x \in \{T, F\}$ unif. rand.

μ

$X_1, \dots, X_k$

$$X_i = \begin{cases} 1 & \text{if } \varphi(i\text{th sample}) = \text{TRUE} \\ 0 & \text{otherwise} \end{cases}$$

$$\Pr[X_1 + \dots + X_k \geq (1+\varepsilon)\mu k] \leq e^{-\varepsilon^2 \mu k/2}$$

$$k = \Omega\left(\frac{1}{\varepsilon^2 \mu}\right)$$

what if $\mu = 2^{-n}$?

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Mean estimation

Input: independent samples of $X \in [0,1]$

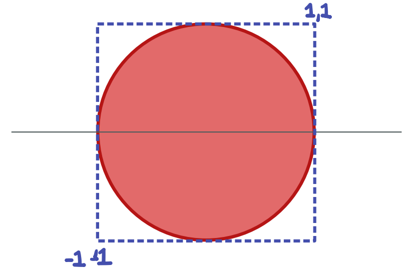
Goal: estimate $E[X]$ (efficiently, w/ few samples)

Sampling approach:

1. Sample $X_1, \ldots, X_k \sim X$
2. return $\frac{1}{k}(X_1 + \cdots + X_k)$

Chernoff: $k = \Omega\left(\frac{1}{\varepsilon^2 E[X]}\right)$

(for $(1\pm\varepsilon)$-APX)



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$P[p \in \bullet] = \frac{\text{Area}(\bullet)}{4}$ $(\approx .785\ldots)$

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$P[X_1 + \cdots + X_k \geq (1+\varepsilon) P[p \in \bullet] n]$

$\leq e^{-\Omega(\varepsilon^2 P[p \in \bullet] k)}$

$\leq \delta/2$ for $k \geq O\left(\frac{\log(1/\delta)}{\varepsilon^2}\right)$

(disjunctive normal form)

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$= (x_1 \wedge \bar{x}_3 \wedge x_5 \wedge \cdots) \vee (x_3 \wedge \bar{x}_4 \wedge \cdots) \cdots$

How many assignments satisfy φ?

$\mu = \underset{x \in \{T,F\} \text{ unif. rand.}}{P}[\varphi(x) = \text{TRUE}] = \left(\begin{matrix}\text{\# satisfying} \\ \text{assignments}\end{matrix}\right) / 2^n$

$X_i = \begin{cases} 1 & \text{if } \varphi(i\text{th sample}) = \text{TRUE} \\ 0 & \text{otherwise} \end{cases}$

$\Pr[X_1 + \cdots + X_k \geq (1+\varepsilon)\mu n] \leq e^{-\varepsilon^2 \mu k / 2}$

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Ideas?

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How many assignments satisfy φ?

let $S_i \subseteq \{T,F\}^n$ = assigments satisfying clause i.

$|S_i| = 2^{n - \#\text{var in clause } i}$

$S_1 \cup S_2 \cup \cdots \cup S_m$ = all satisfying assignments

let $\mu = \dfrac{|S_1 \cup S_2 \cup \cdots \cup S_m|}{|S_1| + |S_2| + \cdots + |S_m|} \geq \dfrac{1}{m}$

$Z = \left\{ \dfrac{1}{\#\ \cancel{\text{sat}}\ \text{clause sat. by assigment}} \right.$

estimate $\mu \Rightarrow$ estimate $|S_1 \cup S_2 \cup \cdots \cup S_m|$

$$E[Z] = \sum_{i=1}^{m} \frac{\cancel{|S_i|}}{D} \sum_{x \in S_i} \frac{1}{\#\text{clauses } x} \cdot \frac{1}{\cancel{|S_i|}}$$

$$= \frac{1}{D} \sum_{x \in N} \frac{1}{\#\text{clauses } x} \cdot \sum_{S_i \ni x} 1$$

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